Ford-Fulkerson Algorithm for Maximum Flow

1. Assign an initial flow f_{ij} (for instance, f_{ij} =0) for all edges

2.Label s by Ø. Mark the other vertices "unlabeled."

3. Find a labeled vertex i that has not yet been scanned . Scan i as follows

For every unlabeled adjacent vertex j, (**a** or **b** or **c**)

a) if $C_{ij} > f_{ij}$ and $f_{ij} \geq 0$

compute $\Delta_{ij} = C_{ij} - f_{ij}$ and Δ_j where

$$\Delta_j = \begin{cases} \Delta ij & \text{if } i = 1 \\ \min(\Delta i,\ \Delta ij) & \text{if } i > 1 \end{cases}$$

Label j with a forward label (i^+, f_{ij})

b) if $C_{ij} > |f_{ij}|$ and $f_{ij} < 0$ (opposite direction)

$\Delta_j = \min(\Delta_j, |f_{ij}|)$

Label j with a backward label (i^-, Δ_j)

c) if $C_{ij} = f_{ij}$ No operation.

If no unlabeled j exists STOP.

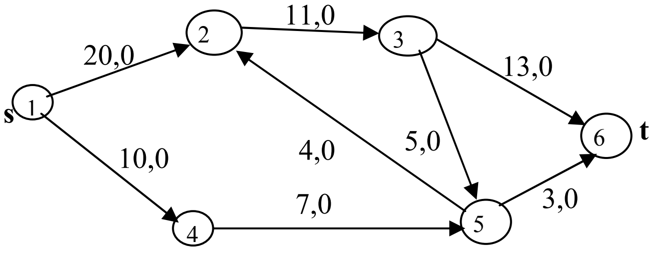
4) Repeat step 3 until t is reached.
[This gives a flow augmenting path P: s —>• t]
If it is impossible to reach t then STOP.

5). Backtrack the path P, using the labels.

6)Using P, augment the existing flow by Δ_t,
Set $f = f + \Delta_t$,.

7) Remove all labels from vertices 2, … , n.
Go to Step 3.

Example: Find the maximum flow from **s** to **t** in the following graph.



Solution 1) C_{12}=20, C_{23}=11, C_{36}=13, C_{35}=5, C_{14}=10, C_{45}=7, C_{52}=4, C_{56}=3,
$f_{12}=f_{23}=f_{36}=f_{35}=f_{14}=f_{45}=f_{52}=f_{56}=0$,

2) vertex 1 (s) is **labeled Ø**, 2,3,4,5,6 are **unlabeled**

3) Scan 1. i=1 ,Adjacent labels 2 and 4. [**j**=2 and **j**=4]

C_{12}=20. f_{12}=0. (perform **a**)

For vertex j=2

$\Delta_{12} = C_{12} - f_{12} = 20 - 0 = 20$

$\Delta_2 = \Delta_{12}$ =20.

L2 = $\{1^+, 20\}$

For vertex j=4

C_{14}=20. f_{14}=0. (perform **a**)

$\Delta_{14} = C_{14} - f_{14} = 10 - 0 = 10$

$\Delta_4 = \Delta_{14}$ =10.

L4 = $\{1^+, 10\}$

Scan 2. i=2 ,Adjacent labels 1, 3 and 5. [**j**=3 and **j**=5]
(j=1 is already labelled)

For vertex j=3

C_{23}=11. f_{23}=0.

$\Delta_{23} = C_{23} - f_{23} = 11 - 0 = 11$

$\Delta_3 = \min(\Delta_2, \Delta_{23}) = \min(20, 11) = 11$

L3 = $\{2^+, 11\}$

For vertex j=5 , $f_{25} < 0$ **(perform b)**

$\Delta_5 = \min(\Delta_2 - |f_{25}|) = \min(20,0) = 0$

L5 = $\{2^-, 0\}$

Scan 3. i=3 ,Adjacent labels 2, 5 and 6. [**j**=6]
(j=2 and j=5 are already labelled)

C_{36}=13. f_{36}=0.

$\Delta_{36} = C_{36} - f_{36} = 13 - 0 = 13$

$\Delta_6 = \min(\Delta_3, \Delta_{36}) = \min(11, 13) = 11$

L6 = $\{3^+, 11\}$

Since **vertex 6 is t** $\boxed{\Delta_t = 11}$

Now all vertices are all labeled

Find the path

L6 = $\{3^+, 11\}$ → **L3** = $\{2^+, 11\}$ → **L2** = $\{1^+, 20\}$ → **L1**

Thus one augmenting path is 1-2-3-6

Add $\mathbf{\Delta_t}$ =11 to this path

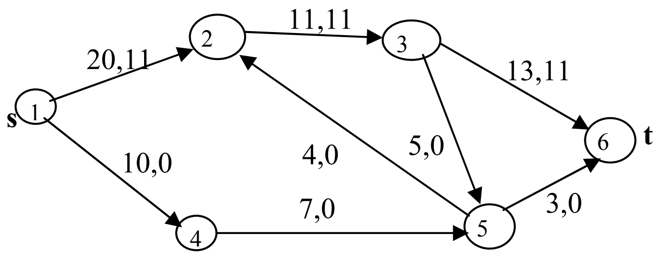
$f_{12(new)} = f_{12(old)} + \mathbf{\Delta_t}$

$f_{12} = 0 + 11$

$f_{23} = 0 + 11 = 11$

$f_{36} = 0 + 11 = 11$

Remove all the labels. Start scanning



1) $C_{12}=20$, $C_{23}=11$, $C_{36}=13$, $C_{35}=5$, $C_{14}=10$, $C_{45}=7$, $C_{52}=4$, $C_{56}=3$,

$f_{12}=f_{23}=f_{36}=11$, $f_{35}=f_{14}=f_{45}=f_{52}=f_{56}=0$,

2) vertex 1 (s) is **labeled Ø**, 2,3,4,5,6 are **unlabeled**

3) Scan 1. i=1 ,Adjacent labels 2 and 4. [**j**=2 and **j**=4]

$C_{12}=20$. $f_{12}=11$. (perform **a**)

For vertex j=2

$\Delta_{12} = C_{12} - f_{12} = 20 - 11=9$

$\Delta_2 = \Delta_{12} =9$.

L2 = {1+, 9}

For vertex j=4

$C_{14}=10$. $f_{14}=0$.

$\Delta_{14} = C_{14} - f_{14} = 10 - 0=10$

$\Delta_4 = \Delta_{14} =10$.

L4 = {1+, 10}

Scan 2. i=2 , Adjacent labels 1, 3 and 5. [**j**=3 and **j**=5]
(j=1 is already labelled)

For vertex j=3,

$C_{23}=11$. $f_{23}=11$. $C_{23} = f_{23}$ **No action.**

For vertex j=5 , **(reverse dir. perform b)**

$\Delta_5 = \min(\Delta_2, |f_{25}|) = \min(9,0) =0$

L5 = {2−, 0}

Scan 3. i=3 **not labeled no action.**

Scan 4. i=4 **No action** Adjacent labels 1, 5.
[**j**=1, **j**=5 are already labeled]

Scan 5. i=5 Adjacent labels 2,3,6,4 [**j**=3, j=6]
(j=2 j=4 are already labelled)

For vertex j=3 **(reverse dir. perform b)**

$\Delta_3 = \min(\Delta_5, |f_{53}|) = \min(0,0) =0$

L3 = {5−, 0}

For vertex 6

$C_{56}=3$. $f_{56}=0$.

$\Delta_{56} = C_{56} - f_{56} = 3 - 0 = 3$

$\Delta_6 = \min(\Delta_5, \Delta_{56}) = \min(0, 3) =0$

L6 = {5+, 0}

Since **vertex 6 is t** Δ_t **=0**

Now all vertices are all labeled

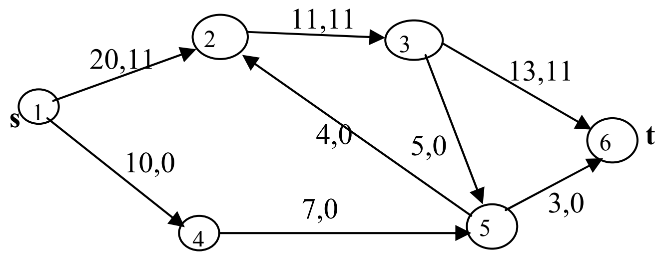
Find the path

L6 = {5+, 3} → **L5** = {2−, 0} → **L2** = {1+, 9} → **L1**

Thus one augmenting path is 1-2-5-6

Add Δ_t =0 to this path. **(No change)**

Remove all the labels. Start scanning



3) Scan 1. i=1

(from above) $\Delta_2 = \Delta_{12} =9$. **L2** = {1+, 9}

$\Delta_4 = \Delta_{14} =7$. **L4** = {1+, 7}

Scan 2: (change path)

Scan 4. i=4

(from above) $\Delta_5 =0$, **L5** = {4+, 10}

Scan 5. i=5

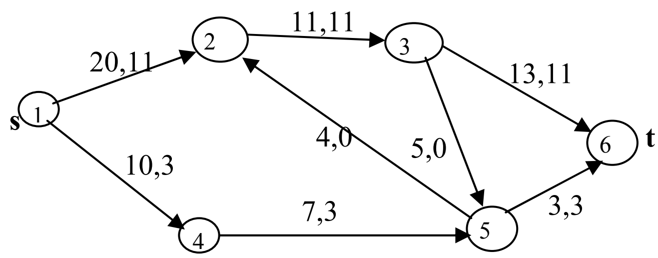
(from above) $\Delta_6 = 3$, **L6** = {5+, 3}

t is reached. $\Delta_6 = \Delta_t =3$

L6 = {5+, 3} → **L5** = {4+, 0} → **L4** = {1+, 9} → **L1**

One augmenting path is 1-4-5-6

Add Δ_t =3 to this path.



Remove all the labels. Start scanning

If we try paths

1,2,5,3,6   1,4,5,3,6 ,   1,4,5,2,3,6

we will get Δ_t =0

Result: We have reached maximum flow.