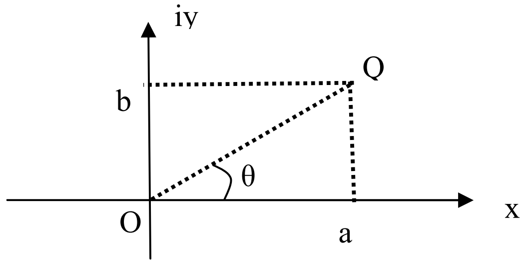
Complex Plane



$z=a+ib$, $|z|=r=\overline{OQ}=\sqrt{a^2+b^2}$, $a=r\cos\theta$, $b=r\sin\theta$

$\sin\theta=\dfrac{b}{r}$, $\cos\theta=\dfrac{a}{r}$, $\tan\theta=\dfrac{b}{a}$,

r: modulus, absolute value, magnitude, amplitude

θ: angle, argument, phase

Polar form of complex numbers.

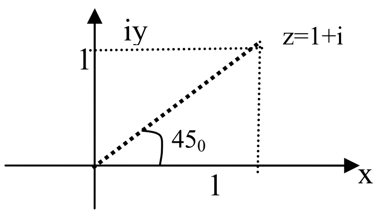
$z = a+bi=r\,e^{j\theta} = r < \theta$

$r=\sqrt{a^2+b^2}$ $\theta=\tan^{-1}\dfrac{b}{a}$

Example CM1: Show the number z=1+i in complex plane, and express it in polar form

Solution: $r=\sqrt{1^2+1^2}=\sqrt{2}=1.41$

$\theta=\tan^{-1}\dfrac{1}{1}=\tan^{-1}1=45^0$



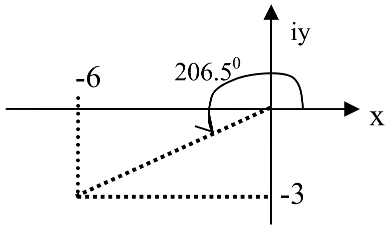
polar form is $z=1.41\,e^{j45(degree)}$

$z=1.41 < 45^0$

Example CM2: Show the number z=-6-3i in complex plane, and express it in polar form

Solution: $r=\sqrt{6^2+3^2}=\sqrt{45}=6.7$

$\theta=\tan^{-1}\dfrac{-3}{-6}=180^0+\tan^{-1}0.5=180^0+26.5^0=206.5^0$



polar form is $z=6.7\,e^{j206.5(degree)}$

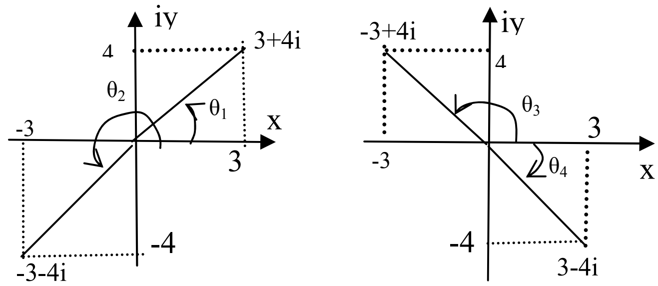
$z=6.7 < 206.5^0$

Note: $\dfrac{-3}{-6}=\dfrac{3}{6}$ **but** $\tan^{-1}\dfrac{-3}{-6}\neq\tan^{-1}\dfrac{3}{6}$

Example CM3: Show the following numbers in complex plane, and express them in polar form

a) $z_1=3+4i$ b) $z_2=-3-4i$ c) $z_3=-3+4i$ d) $z_4=3-4i$

Solution:



a) $z_1=3+4i$

$\theta_1=\tan^{-1}\dfrac{4}{3}=\tan^{-1}1.33=53.13^0$

$r_1=\sqrt{3^2+4^2}=\sqrt{25}=5$

Polar form $z_1=5\,e^{j53.1(degree)}=5 < 53.13^0$

b) $z_2=-3-4i$

$\theta_2=180+\tan^{-1}\dfrac{4}{3}=180+\tan^{-1}1.33=180^0+53.1^0=233.1^0$

$r_2=\sqrt{3^2+4^2}=\sqrt{25}=5$

Polar form $z_2=5\,e^{j233.1(degree)}=5 < 233.1^0$

c) $z_3=-3+4i$

$\theta_3=180-\tan^{-1}\dfrac{4}{3}=180-\tan^{-1}1.33=180^0-53.1^0=126.9^0$

$r_3=\sqrt{3^2+4^2}=\sqrt{25}=5$

Polar form $z_3=5\,e^{j126.9(degree)}\ 5\,e^{j223.1(degree)}=5 < 126.9^0$

d) z1=3-4i

$\theta_4=-\tan^{-1}\dfrac{4}{3}=-\tan^{-1}1.33=-53.13^0$

$r_4=\sqrt{3^2+4^2}=\sqrt{25}=5$

Polar form $z_4=5\,e^{-j223.1(degree)}=5\,e^{-j53.1(degree)}=5<-53.1^0$

Euler Formula: $e^{ix}=\cos x+i\sin x$

$e^{i60(degree)}=\cos 60^0+i\sin 60^0=0.5+0.866\,i$

$e^{i170(degree)}=\cos 170^0+i\sin 170^0=-0.98+0.17\,i$

$e^{i220(degree)}=\cos 220^0+i\sin 220^0=-0.76-0.64\,i$

$e^{i335(degree)}=\cos 335^0+i\sin 335^0=0.9-0.42\,i$

$e^{i2(radian)}=\cos 2(radian)+i\sin 2\,(radian)=-0.41+0.9\,i$

$e^{i2}=\cos 2+i\sin 2=-0.41+0.9\,i$

$e^{i17}=\cos 17+i\sin 17=-0.27-0.96\,i$

e^{-i2}=cos (-2) + i sin (-2) = -0.41 - 0.9 i
e^{-i17}=cos (-17) + i sin (-17) = -0.27 + 0.96 i

Multiplication, division, power

Example CM5- Calculate $\frac{3+4i}{-4+6i}$

Method I.

$$\frac{3+4i}{-4+6i}=\frac{(3+4i)(-4-6i)}{(-4+6i)(-4-6i)}=\frac{-12-18i-16i+24}{4^2+6^2}$$

$$=\frac{12-34i}{52}=0.23-0.65i$$

Method II: | 3+4i |= $\sqrt{3^2+4^2}$ =5

<(3+4i)=$\tan^{-1}$ (4/3)=53.1^0

| -4+6i |= $\sqrt{4^2+6^2}$ =7.21

<(-4+6i)= $\tan^{-1}\frac{6}{-4}$ = 180 - $\tan^{-1}\frac{6}{4}$ =180-56.3=123.7^0

Control: $7.21e^{i123.7(degree)}$=7.21(cos 123.7^0+ i sin 123.7^0)
=7.21(-0.55 + i 0.83)
=7.21(-0.55 + i 0.83)
=- 3.9 + 5.9 i ≈ -4 +6 i

$$\frac{3+4i}{-4+6i}=\frac{5\ e^{i53.1^0}}{7.21\ e^{i123.7^0}}=\frac{5}{7.21}e^{i(53.1-123.7)}=0.69\,e^{i(-70.6)}$$

$$=0.69(\cos(-70.6)+i\sin(-70.6))=0.69(0.33-i\,0.94)$$

= 0.23 - 0.65i

Example CM6- It is given that

P=(-6+10i)(-3+i)(10-4i)(7+3i)
Q=(5+3i)(3-7i)(-5+2i)(-9+3i)

Calculate

$$A=\frac{P}{Q}=\frac{(-6+10i)(-3+i)(10-4i)(7+3i)}{(5+3i)(3-7i)(-5+2i)(-9+3i)}$$

Solution: Method 1:

(-6+10i)(-3+i)=(-6)(-3)-6i-30i+$10i^2$ =18-10-6i-30i
=8-36i
(10-4i)(7+3i)= 82+2i
P=(8-36i)(82+2i)= 728-2936i

(5+3i)(3-7i)= 36-26i
(-5+2i)(-9+3i)= 39-33i
Q=(36-26i)(39-33i)= 546-2202i
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$$A=\frac{(728-2936i)}{(546-2202i)}=\frac{(728-2936i)(546+2202i)}{(546-2202i)(546+2202i)}$$

$$\frac{6862560+0i}{5146920}=1.3333+0i$$

Method 2:

Numerator

-6+10i=$\sqrt{6^2+10^2}\ e^{i\tan^{-1}\frac{10}{-6}}$=11.66 e^{121i}
-3+i=3.16 e^{161i}
10-4i=10.77 $e^{-21.8i}$
7+3i=7.61 $e^{23.2i}$
Amp=11.66 *x* 3.16 *x* 10.77 *x* 7.61=3019.8
Angle=121+161+(-21.8)+23.2=283.4
Denominator
5+3i=5.83 $e^{30.9i}$
3-7i=7.61 $e^{-66.8i}$
-5+2i =5.38 e^{158i}
-9+3i 9.48 e^{161i}
Amp=5.83 7.61 5.38 9.48=2262.7
Angle=30.9+(-66.8)+(158)+161=283.1

Amplitudes are divided angles are subtracted the

$$A=\frac{3019.8\,e^{283.4i}}{2262.7\ e^{283.1i}}=\frac{3019.8}{2262.7}\ e^{(283.4-283.1)i}$$

=1.33 $e^{0.3i}$ = 1.33 (cos 0.3 +i sin 0.3)
=1.329+0.0069i

Method 1 and method2 gave same results. The difference is due to truncation error.

Example CM2- q=3-4i calculate a) q^2 b) q^7.

Solution: 3-4i=$\sqrt{3^2+4^2}\ e^{i\tan^{-1}\frac{-4}{3}}$ =5 $e^{-i53.13}$

a) $(3-4i)^2$= $(5\ e^{-i53.13})^2$=5^2 $e^{2(-i53.13)}$= 25 $e^{-i106.26}$
=25(25*(-0.279 -i 0.960)=
=-6.999 - i 24.00025

b) $(3-4i)^7$=$(5\ e^{-i53.3})^7$=5^7 $e^{7(-i53.3)}$= 78125 $e^{-i371.9}$
=78125 [cos(-371.9)+ i sin(-371.9)]
=78125 [cos(-371.9)+ i sin(-371.9)]
=78125(0.97-i 0.22)
=76122.6- i 17574.1

Note:

$(3-4i)^2$=(3-4i) (3-4i)=-7-24i but we have found above that $(3-4i)^2$=-6.999 - i 24.00025 The difference is due to the **truncation** error. we assumed $\tan^{-1}$(-4/3)=-53.13 in fact $\tan^{-1}$(-4/3)=-53.13010235…