Cauchy Rieman Equation: $U_X=V_Y$, $U_Y=-V_X$

If $U_X=V_Y$ $U_Y=-V_X$ $\rightarrow U(x,y)+iV(x,y)$ is analytic

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Exmple CF1

Show that the function $f(z)=z^2+z$ is analytic everywhere

$z=x+iy$,

$f(z)=z^2+z=(x+iy)^2+z$

$= x^2+(iy)^2+2xiy+x+iy = x^2+i^2y^2+2xiy+x+iy$

$= x^2-y^2+x+2xiy+iy = (x^2-y^2+x)+i(2xy+y)$

$= U(x,y)+iV(x,y)$

$U(x,y)=x^2-y^2+x$

$V(x,y)=2xy+y$

$\frac{dU}{dx}=U_X=2x+1$ $\frac{dU}{dy}=U_Y=-2y$

$\frac{dV}{dx}=V_X=2y$ $\frac{dV}{dy}=V_Y=2x+1$

Since $U_X=V_y$ and $U_Y=-V_Y$ the function is **analytic**.

The equality $U_X=V_y$ and $U_Y=-V_Y$ holds everywhere so the functiojn is **analytic everywhere**.

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Example CF2

Show that the function $f(z)=z\,e^{-Z}$ is analytic everywhere

$z=x+iy$, $f(z)=z\,e^{-Z}=(x+iy)\,e^{-(x+iy)}$

$e^{-(x+iy)}=e^{-x}\,e^{-iy}=e^{-x}\,[\cos(-y)+i\sin(-y)]$

Note: $\cos(-y)=\cos(y)$ and $\sin(-y)=-\sin(y)$

Thus $e^{-(x+iy)}=e^{-x}\,[\cos(y)-i\sin(y)]$

$f(z)=(x+iy)\,e^{-(x+iy)}=(x+iy)\,e^{-x}\,[\cos(y)-i\sin(y)]$

$= x\,e^{-x}[\cos y - i\sin y]+iy\,e^{-x}[\cos y - i\sin y]$

$= x\,e^{-x}\cos y - i\,x\,e^{-x}\sin y + iy\,e^{-x}\cos y - i^2 y\,e^{-x}\sin y$

$= x\,e^{-x}\cos y + y\,e^{-x}\sin y - i\,x\,e^{-x}\sin y + iy\,e^{-x}\cos y$

$= x\,e^{-x}\cos y + y\,e^{-x}\sin y + i\,[-x\,e^{-x}\sin y + y\,e^{-x}\cos y]$

$= U(x,y) + i\,V(x,y)$

$U(x,y)= x\,e^{-x}\cos y + y\,e^{-x}\sin y$

$V(x,y)= -x\,e^{-x}\sin y + y\,e^{-x}\cos y$

Note: $\frac{d}{dx}(f\,g)=\frac{df}{dx}\,g+f\,\frac{dg}{dx}$

$\frac{d}{dx}(x\,e^{-x})=1\,(e^{-x})+x\,(-e^{-x})=e^{-x}-x\,e^{-x}$

$\frac{d}{dy}(y\sin y)=\sin y+y\cos y$

$\frac{d}{dy}(y\cos y)=\cos y+y\,(-\sin y)$

$U_X=\frac{d}{dx}(x\,e^{-x}\cos y+y\,e^{-x}\sin y)$

$=\frac{d}{dx}(x\,e^{-x}\cos y)+\frac{d}{dx}(y\,e^{-x}\sin y)$

$= e^{-x}\cos y - x\,e^{-x}\cos y - e^{-x}\,y\sin y$

$U_Y= -x\,e^{-x}\sin y + e^{-x}\sin y + e^{-x}\,y\cos y$

$V_X= x\,e^{-x}\sin y - e^{-x}\sin y - e^{-x}\,y\cos y$

$V_Y = e^{-x}\cos y - x\,e^{-x}\cos y - e^{-x}\,y\sin y$

Since $U_X=V_Y$ and $U_Y=-V_X$ the function is **analytic**.

.**Example CF3**

Determine whether the function $f(z)=z^2+iz$ is analytic or not?

Solution: $f(z)=z^2+iz=(x+iy)^2+i\,(x+iy)=x^2-y^2-y+i(2xy+x)$

$= U(x,y)+iV(x,y)$

$U(x,y)=x^2-y^2-y$

$V(x,y)=2xy+x$

$U_X=2x$ $U_Y=-2y-1$ $V_X=2y+1$ $V_Y=2x$

$U_X=V_y$ $U_Y=-V_X$ the function is **analytic.**

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Example CF4

Determine whether the function $f(z)=3\bar{z}+1$ is analytic or not? ($\bar{z}$:is complex conjugate of z)

Solution:

$f(z)=3\bar{z}+1=3\,\overline{(x+iy)}+1=3(x-iy)+1=3x-i3y+1$

$U(x,y)=3x+1$

$V(x,y)=-3y$

$U_X=3$ $U_Y=0$ $V_X=0$ $V_Y=\mathbf{-3}$.

$U_X\neq V_y$ the function is **not analytic**.

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Exercise CF5

Determine whether the function $f(z)=e^{-\bar{z}}$ is analytic or not? ($\bar{z}$:is complex conjugate of z)

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Laplace Equation:

If $f(z)=U(x,y)+iV(x,y)$ is analytic then

$U_{XX}+U_{YY}=0$ $V_{XX}+V_{YY}=0$

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Example CF6

$f(z)=z^3$

a) Calculate U(x,y) and V(x,y)
b) Show that this function is analytic
c) Show that this function satisfies **Laplace** equation

Solution

$f(z)=z^3=(x+iy)^3=x^3-3xy^2+i(-y^3+3x^2y)$

a) $U(x,y)=x^3-3xy^2$ $V(x,y)=-y^3+3x^2y$

b) $U_X=3x^2-3y^2$, $U_Y=-6xy$, $V_X=6xy$, $V_Y=3x^2-3y^2$

$U_X=V_y$, $U_Y=-V_X$ $\rightarrow$ the function is **analytic**.

c) $U_{XX}=6x$, $U_{YY}=-6x$, $V_{XX}=6y$, $V_{YY}=-6y$

$U_{XX}+U_{YY}=6x-6x=0$

$V_{XY}+V_{YY}=6y-6y=0$