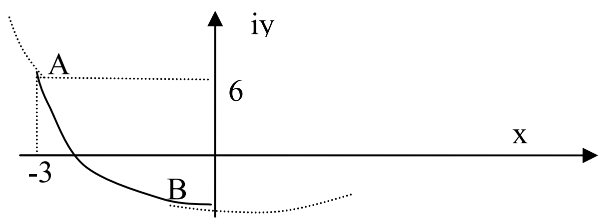
$$\oint_C f(z)dz = \int_a^b f[z(t)]\frac{dz}{dt}dt$$

$$z = x+iy \;\rightarrow\; \frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt}, \quad f(z)=u(x,y)+iv(x,y)$$

$$\oint_C f(z)dz = \int_a^b (u+iv)\left(\frac{dx}{dt} + i\frac{dy}{dt}\right)dt$$

Example problem Calculate $\oint_C \frac{1}{z^5+4z^2+1}dz$

Where C is the curve $x^2-x-6-y=0$ from A=-3+6i to B=-1-4i



Solution:

$$f(z) = \frac{1}{z^5+4z^2+1} \qquad z=x+iy$$

The curve equation is: $x^2-x-6-y=0$ or $y=x^2-x-6$.
Define x=t $\rightarrow y(t)=t^2-t-6$ and

$$\frac{dx}{dt} = 1, \qquad \frac{dy}{dt} = 2t-1$$

Borders: x=t point A→t=-3, point B→t=-1

$$f(z) = \frac{1}{z^5+4z^2+1} = \frac{1}{(x+iy)^5+4(x+iy)^2+1}$$

$$= \frac{1}{\left[t+i(t^2-t-6)\right]^5 + 4\left[t+i(t^2-t-6)\right]^2+1}$$

$$\frac{dx}{dt} + i\frac{dy}{dt} = 1+i(2t-1)$$

$$\oint_C f(z)dz = \int_{-3}^{-1}\frac{1}{\left[t+i(t^2-t-6)\right]^5+4\left[t+i(t^2-t-6)\right]^2+1}\left[1+i(2t-1)\right]dt$$

$$\oint_C f(z)dz =$$

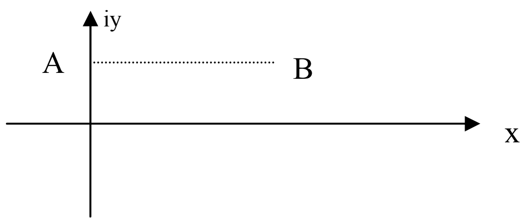
$$\int_{-3}^{-1}\frac{\left[1+i(2t-1)\right]}{\left[t+i(t^2-t-6)\right]^5+4\left[t+i(t^2-t-6)\right]^2+1}dt$$

We converted complex integral into line integral.

---.

Example CI2: Calculate $\oint_C z^2dz$ where C is a straight line from A=i to B=1+i

Solution



The straight line from A=i to B=i+2 is a parallel line to the x axis. y is constant(y=1). thus dy=0.
z=x+iy=x+2i → dz=dx

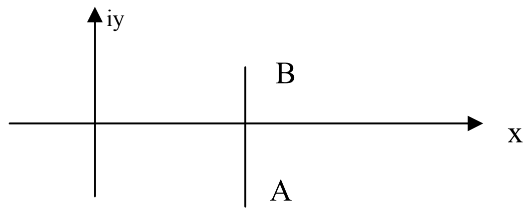
$$\oint_C f(z)dz = \int_{x=0}^{x=1}(x+i)^2\,dx = \frac{1}{3}(x+i)^3\Big|_{x=0}^{x=1} = 0.66+4i$$

Note: $\int (x+a)^2\,dx = \frac{1}{3}(x+a)^3$

--..

Example CI3: Calculate $\oint_C z^2dz$, where C is a straight line from A=1-i to B=1+3i

Solution



The straight line from A=1-i to B=1+3i is a parallel line to the y axis. x is constant(x=1). thus dx=0.
z=x+iy=1+iy → dz=i dy

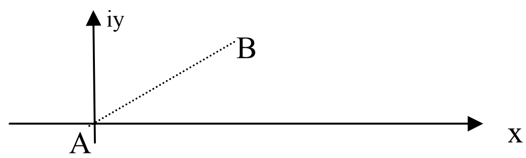
$$\oint_C f(z)dz = \int_{y=-1}^{y=3}(1+iy)^2\,i\,dy = i\frac{1}{3}(1+iy)^3\Big|_{yx=-1}^{y=3} = -8-5.33i$$

Note: $\int (a+bx)^2\,dx = \frac{1}{3b}(a+bx)^3$

---.

Example CI4: Calculate $\oint_C z^2dz$, where C is a straight line y=2x from A=0 to B=1+2i

Solution



x=t →dx=dt   y=2x=2t , dy=2dt

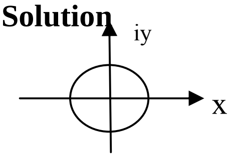
$$\oint_C f(z)dz = \int_{t=0}^{t=1}(x+iy)^2\,(dx+i\,dy) = \int_{t=0}^{t=1}(t+i2t)^2\,(dt+i\,2dt)$$

$$= \int_{t=0}^{t=1}[(1+2i)t]^2\,(1+2i)dt = (1+2i)^3\int_{t=0}^{t=1}t^2\,dt = -11-2i$$

---..

Example CI5: Calculate $\oint_C z^2dz$, where C is the unit circle

Solution



$z=r\,e^{i\Theta}=e^{i\Theta}$ $dz=d\Theta$

$$\oint_C z^2\,dz = \int_{\theta=0}^{\theta=2\pi}(e^{i\theta})^2 e^{i\theta}d\theta = 0$$

Example CI6: $\oint_C \frac{1}{z}dz = 2\pi i$, where C is the unit circle