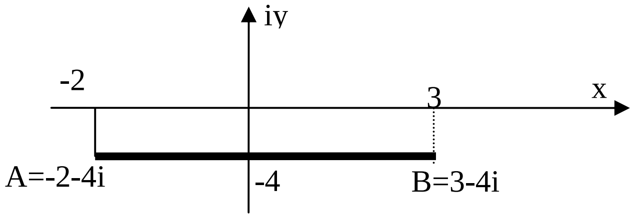
Example AE-18. Evaluate $\int_C z^2\, dz$ from A=-2-4i to B=3-4i along the straight line. C is AB line.



Solution:

$$\int_C f(z)\, dz = \int_C z^2\, dz$$

Replace z=x+iy, dz=dx+idy

$$\int_C f(x+iy)\,(dx+idy) = \int_C (x+iy)^2\,(dx+idy)$$

The straight line from A=-2-4i to B=3-4i is parallel to the x axis. Thus y=-4 is constant.

y=-4 ==> dy=0

x varies from **(-2) to (+3)**

replace y=-4 and dy=0

$$\int_C f(x+iy)\,(dx+idy) = \int_{x=-2}^{3} \left(x + i\,(-4)\,\right)^2\,(dx + i\,0)$$

$$= \int_{x=-2}^{3} (x-4i)^2\;\; dx$$

$$= \int_{x=-2}^{3} (x^2 - 8i\,x - 16)\;\, dx$$

$$= \left(\frac{x^3}{3} - 8i\,\frac{x^2}{2} - 16x\right)\Bigg|_{x=-2}^{3}$$

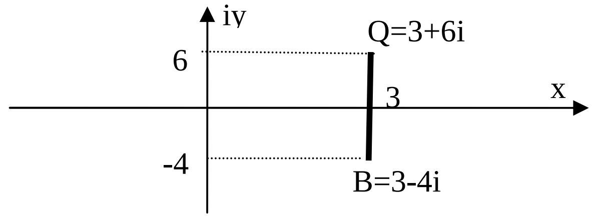
$$= \left(\frac{3^3}{3} - 8i\,\frac{3^2}{2} - 16\,(3)\right) - \left(\frac{(-2)^3}{3} - 8i\,\frac{(-2)^2}{2} - 16\,(-2)\right)$$

=(9 – 36i – 48) – (-2.66 – 16i +32)

=(– 39 – 36i) – (-29.33 – 16i)

= – 68.3 – 20 i

Example AE-19. Evaluate $\int_C z^2\, dz$ from B=3-4i to Q=3+6i along the straight line. C is BQ line.



Solution:

$$\int_C f(z)\, dz = \int_C z^2\, dz$$

Replace z=x+iy, dz=dx+idy

$$\int_C f(x+iy)\,(dx+idy) = \int_C (x+iy)^2\,(dx+idy)$$

The straight line from B=3-4i to Q=3+6i is parallel to the y axis. Thus x=3 is constant.

x=3 ==> dx=0

y varies from **(-4) to (+6)**

It is **not** from **(-4i) to (+6i)**

replace x=3 and dx=0

$$\int_C f(x+iy)\,(dx+idy) = \int_{y=-4}^{6} (3+iy)^2\,(0+idy)$$

$$= \int_{y=-4}^{6} (3+iy)^2\; i\,dy$$

$$= \int_{y=-4}^{6} (9 + 6i\,y + i^2 y^2)\, i\,dy$$

$$= i\int_{y=-4}^{6} (-y^2 + 6i\,y + 9)\;\, dy$$

$$= i\left[\;\left(-\frac{y^3}{3} + 6i\,\frac{y^2}{2} + 9y\right)\Bigg|_{-4}^{6}\;\right]$$

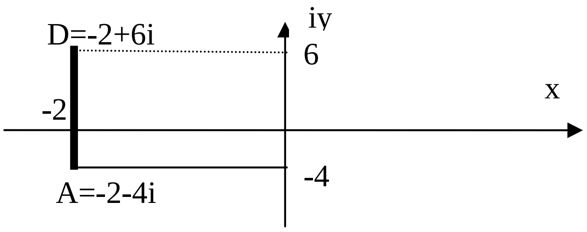
$$= i\left[\left(-\frac{6^3}{3} + 6i\,\frac{6^2}{2} + 9(6)\right) - \left(-\frac{(-4)^3}{3} + 6i\,\frac{(-4)^2}{2} + 9(-4)\right)\right]$$

$$= i\left[(-72 + 108i + 54) - (-21.33 + 48i - 36)\right]$$

=i [(-18 + 108i) – (-14.6+48i)]

= – 60 – 3.3i

Example AE-20. Evaluate $\int_C z^2\,dz$ from A=-2-4i to B=-2+6i along the straight line. C is AD line.



Solution:
The straight line from A=-2-4i to D=-2+6i is parallel to the y axis. Thus x=-2 is constant. dx=0.

$$\int_C f(x+iy)\,(dx+idy) = \int_{y=-4}^{6} (-2+iy)^2\,(0+i\,dy)$$

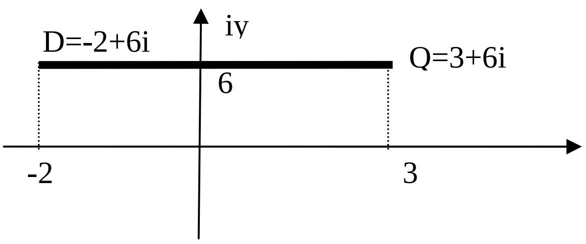
$$= i\int_{y=-4}^{6} (-2+iy)^2\,dy \quad = \int_{y=-4}^{6}\left(4-4iy+y^2\right)dy$$

$$= i\left(4y - (1/2)4\,y^2 - (1/3)y^3\right)\Big|_{y=-4}^{6}$$

$$= i\left(4y - 2\,y^2 - 0.33y^3\right)\Big|_{y=-4}^{6}$$

= 40-53.3i

Example AE-21. Evaluate $\int_C z^2\,dz$ from D=-2+6i to Q=3+6i along the straight line. C is DQ line.



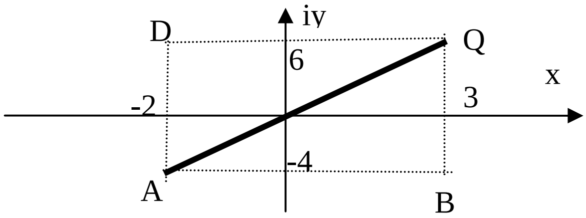
Solution:
The straight line from D=-2+6i to Q=3+6i is parallel to the x axis. Thus y=6 is constant. dy=0.

$$\int_C f(x+iy)\,(dx+idy) = \int_C (x+i6)^2\,(dx+i\,0)$$

$$= \int_{x=-2}^{3}(x+i6)^2\,dx = \int_{x=-2}^{3}(x^2+12ix-36)dx$$

$$\left(\frac{x^3}{3}+12i\frac{x^2}{2}-36x\right)\Bigg|_{-2}^{3} = -168.3+30i$$

Example AE-22. Evaluate $\int_C z^2\,dz$ from A=-2-4i to Q=3+6i along the straight line y=2x. C is AQ line.



Solution: We can use parametric equations

Define x=t ==> dx=dt

y=2x ==> y=2t ==>dy=2dt

Integration starts at x=-2 ==> t= -2

Integration ends at x=3 ==> t= 3

$$\int_C z^2 dz = \int_C (x+iy)^2\,(dx+idy)$$

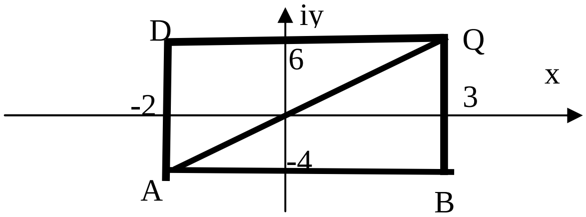
$$= \int_C (t+i\,(2t)\,)^2\,(dt+i\,2\,dt)$$

$$= \int_C [(1+2i)t]^2\,(1+2i)dt = \int_C (1+2i)^3\,t^2\,dt$$

$$= (1+2i)^3\int_C t^2\,dt = (1+2i)^3\,\frac{t^3}{3}\Bigg|_{t=-2}^{t=3}$$

$$= (1+2i)^3\,\frac{1}{3}\left(3^3 - (-2)^3\right) = -128.3-23.3i$$

Summary



$$\int_{AB} z^2\,dz = -68.3-20i \qquad \int_{BQ} z^2\,dz = -60-3.3i$$

$$\int_{AD} z^2\,dz = 40-53.3i \qquad \int_{DQ} z^2\,dz = -168.3+30i$$

$$\int_{ABQ} z^2\,dz = \int_{AB} z^2\,dz + \int_{BQ} z^2\,dz = (-68.3-20)+(-60-3.3)$$

= -128.3-23.3i

$$\int_{ADQ} z^2\,dz = \int_{AD} z^2\,dz + \int_{DQ} z^2\,dz = (40-53.3i)+(-168.3+30i)$$

= -128.3-23.3i

$$\int_{ACQ} z^2\,dz = \int_{ADQ} z^2\,dz = \int_{AQ} z^2\,dz$$ **All are equal**