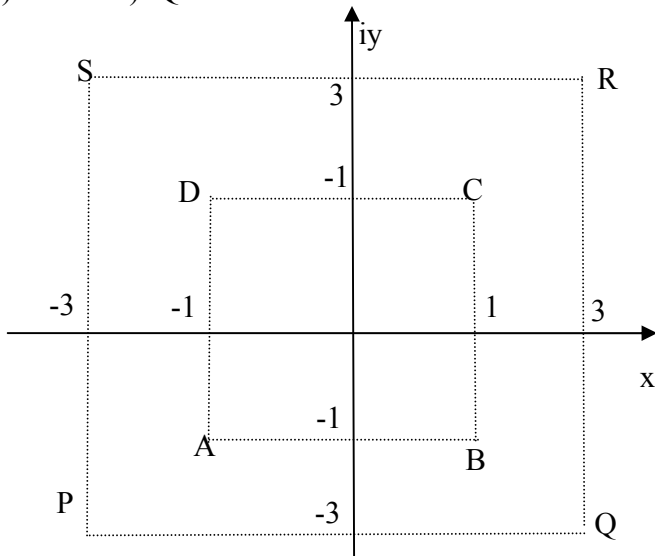


Example AE-18. Evaluate $\int_C \frac{1}{z-2} dz$ over the path

a) ABCD b) PQRS



Solution: a) Along the line AB $y=-1$ constant. $dy=0$

$$\begin{aligned} \int_{AB} \frac{1}{z-2} dz &= \int_{AB} \frac{1}{x+iy-2} (dx + i dy) = \int_{x=-1}^1 \frac{1}{x-i-2} dx \\ &= \left. \ln(x-2-i) \right|_{-1}^1 = \left(\ln \sqrt{(x-2)^2 + 1} + i \tan^{-1} \left(\frac{-1}{x-2} \right) \right) \Big|_{-1}^1 \\ &= -0.804 + 0.46i \end{aligned}$$

Along the line BC $x=1$ constant. $dx=0$

$$\begin{aligned} \int_{AB} \frac{1}{z-2} dz &= \int_{AB} \frac{1}{x+iy-2} (dx + i dy) = \int_{y=-1}^1 \frac{1}{1+iy-2} i dy \\ &= \int_{y=-1}^1 \frac{1}{iy-1} i dy = \int_{y=-1}^1 \frac{\frac{i}{y-\frac{1}{i}}}{y-\frac{1}{i}} i dy = \int_{y=-1}^1 \frac{-i}{y+i} i dy \\ &= \int_{y=-1}^1 \frac{1}{y+i} dy = \left. \ln(y+i) \right|_{-1}^1 = \left(\ln \sqrt{y^2 + 1} + \tan^{-1} \frac{1}{y} \right) \Big|_{-1}^1 \\ &= -1.57i \end{aligned}$$

Along the line CD $y=1$ constant. $dy=0$

$$\begin{aligned} \int_{AB} \frac{1}{z-2} dz &= \int_{AB} \frac{1}{x+iy-2} (dx + i dy) = \int_{x=1}^{-1} \frac{1}{x+i-2} dx \\ &= 0.804 + 0.46i \end{aligned}$$

Along the line DA $x=-1$, $dx=0$

$$\int_{DA} \frac{1}{z-2} dz = \int_{y=1}^{-1} \frac{1}{-1+iy-2} dy = 0.6435i$$

Integration along ABCD

$$\begin{aligned} \int_{ABCD} \frac{1}{z-2} dz &= \int_{AB} \frac{1}{z-2} dz + \int_{BC} \frac{1}{z-2} dz + \int_{CD} \frac{1}{z-2} dz + \int_{DA} \frac{1}{z-2} dz \\ \int_{ABCD} \frac{1}{z-2} dz &= (-0.804 + 0.46i) + (-1.57i) + (0.804 + 0.46i) \\ &\quad + (0.6435i) = 0 \end{aligned}$$

b) **Along the line PQ** $y=-3$ constant. $dy=0$

$$\begin{aligned} \int_{PQ} \frac{1}{z-2} dz &= \int_{PQ} \frac{1}{x+iy-2} (dx + i dy) = \int_{x=-3}^3 \frac{1}{x-3i-2} dx \\ &= -0.6119 + 1.3521i \end{aligned}$$

Along the line QR $x=3$ constant. $dx=0$

$$\begin{aligned} \int_{AB} \frac{1}{z-2} dz &= \int_{y=-3}^3 \frac{1}{3+iy-2} i dy = \int_{y=-3}^3 \frac{1}{y-i} dy \\ \ln(y-i) \Big|_{-3}^3 &= 2.498i \end{aligned}$$

Along the line RS $y=3$ constant. $dy=0$

$$\begin{aligned} \int_{PQ} \frac{1}{z-2} dz &= \int_{x=3}^{-3} \frac{1}{x-3i-2} dx = \\ &= 0.6119 + 1.3521i \end{aligned}$$

Along the line SP $x=-3$ constant. $dx=0$

$$\int_{y=3}^{-3} \frac{1}{-3+iy-2} i dy = 1.088i$$

$$\begin{aligned} \int_{PQRS} \frac{1}{z-2} dz &= \int_{PQ} \frac{1}{z-2} dz + \int_{QR} \frac{1}{z-2} dz + \int_{RS} \frac{1}{z-2} dz + \int_{SP} \frac{1}{z-2} dz \\ &= (-0.611 + 1.35i) + 2.49i + (0.611 + 1.35i) + 1.088i \\ &= 6.28i = 2\pi i \end{aligned}$$

Result

$$\int_{ABCD} \frac{1}{z-2} dz = 0 \quad \text{but} \quad \int_{PQRS} \frac{1}{z-2} dz \neq 0$$

WHY?