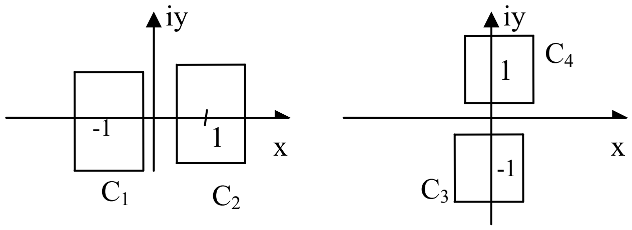
Cauchy Integral Theorem $\oint_C \frac{f(z)}{z-z_0}dz = 2\pi\, i\, f(z_0)$

Example IC61. Calculate $\oint_C \frac{z^2+3}{z-1}dz$ over C_1, C_2, C_3, C_4



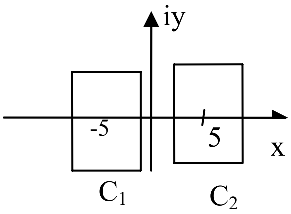
Solution: C_1, C_3, C_4 do not enclose z=1. Thus

$$\oint_{C_1} \frac{z^2+3}{z-1}dz = 0, \quad \oint_{C3} \frac{z^2+3}{z-1}dz = 0, \quad \oint_{C4} \frac{z^2+3}{z-1}dz = 0$$

For C_2 $z_0=1$, $f(z)=z^2+3$, $f(z_0)= f(1)=1^2+3=4$

$$\oint_C \frac{z^2+3}{z-1}dz = 2\pi\, i\, f(z_0) = 2\pi\, i\, 4 == 8\pi\, i = 25.13i$$

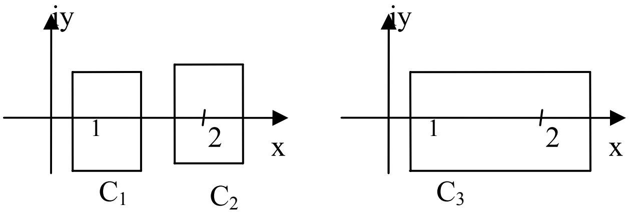
Example IC62. Calculate $\oint_C \frac{2}{z-5}dz$ over C_1, C_2



Integration over C_1 is zero. $\oint_{C1} \frac{2}{z-5}dz = 0$

For C_2 $z_0=5$, $f(z0)=2$. $\oint_{C2} \frac{2}{z-5}dz = 2\pi\, i\, 2 = 12.56i$

Example IC63: Calculate $\oint_C \frac{z+6}{(z-1)(z-2)}dz$ over C_1, C_2, C_3



Solution: $\frac{z+6}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} = \frac{A(z-2)+B(z-1)}{(z-1)(z-2)}$

$$\frac{z+6}{(z-1)(z-2)} = \frac{A(z-2)+B(z-1)}{(z-1)(z-2)} = \frac{(A+B)z-2A-B}{(z-1)(z-2)}$$

A+B=1 -2A-B=6 ==> A=-7 B=8

Integration over C_1

$$\oint_{C1} \frac{z+6}{(z-1)(z-2)}dz = \oint_{C1} \frac{A}{z-1}dz + \oint_{C1} \frac{B}{z-2}dz$$

C_1 encloses the point z=1.

$$\oint_{C1} \frac{A}{z-1}dz = 2\pi i\, A = 2\pi i\,(-7) = -14\pi i = -43.98i$$

$\oint_{C1} \frac{B}{z-2}dz = 0$ Because C_1 does not enclose z=2

Thus

$$\oint_{C1} \frac{A}{z-1}dz + \oint_{C1} \frac{B}{z-2}dz = -43.98i + 0 = -43.98i$$

Result

$$\oint_{C1} \frac{z+6}{(z-1)(z-2)}dz = -43.98i$$

Integration over C_2

$$\oint_{C2} \frac{z+6}{(z-1)(z-2)}dz = \oint_{C2} \frac{A}{z-1}dz + \oint_{C2} \frac{B}{z-2}dz$$

$\oint_{C2} \frac{A}{z-1}dz = 0$ because C_2 does not enclose z=1

$$\oint_{C2} \frac{B}{z-2}dz = 2\pi i\, B = 2\pi i\,(8) = 16\pi i = 50.26i$$

Result

$$\oint_{C2} \frac{z+6}{(z-1)(z-2)}dz = 50.26i$$

Integration over C_3

$$\oint_{C3} \frac{z+6}{(z-1)(z-2)}dz = \oint_{C3} \frac{A}{z-1}dz + \oint_{C3} \frac{B}{z-2}dz$$

$$\oint_{C3} \frac{A}{z-1}dz = 2\pi i\, A = 2\pi i\,(-7) = -14\pi i = -43.98i$$

$$\oint_{C3} \frac{B}{z-2}dz = 2\pi i\, B = 2\pi i\,(8) = 16\pi i = 50.26i$$

$$\oint_{C3} \frac{z+6}{(z-1)(z-2)}dz = \oint_{C3} \frac{A}{z-1}dz + \oint_{C3} \frac{B}{z-2}dz = -43.98i + 50.26i$$

$$= 6.28i$$

Notes on partial fraction

$$\frac{m}{(z+a)(z+b)}=\frac{A}{z+a}+\frac{B}{z+b}$$

$$\frac{z+m}{(z+a)(z+b)}=\frac{C}{z+a}+\frac{D}{z+b}$$

$$\frac{z^2+mz+n}{(z+a)(z+b)}=E+\frac{F}{z+a}+\frac{G}{z+b}$$

$$\frac{z^3+mz^2+nz+q}{(z+a)(z+b)}=Hz+K+\frac{L}{z+a}+\frac{J}{z+b}$$

and

$$\frac{z+m}{z+a}=A+\frac{B}{z+a}$$

Example IC64. Calculate $\oint_C \frac{z+8}{z+3}dz$ where C encloses the point z=-3.

Solution 1: Use Cauchy integral formula directly

$z_0=-3$, $f(z)=z+8$, $f(z_0)=-3+8=5$

$$\oint_C \frac{z+8}{z+3}dz = 2\pi i\,(5)=10\pi i=31.4i$$

Solution 2: $\frac{z+8}{z+3}=A+\frac{B}{z+3}=\frac{A(z+3)}{z+3}+\frac{B}{z+3}$

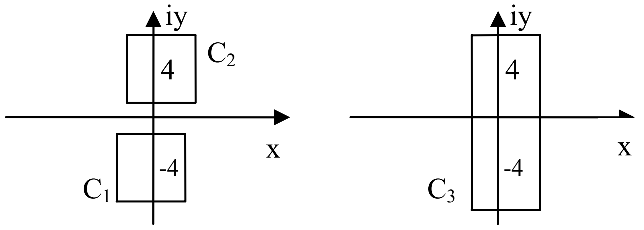
Z+8=Az+3A+B ==> A=1, B=5

$$\oint_C \frac{z+8}{z+3}dz=\oint_C A\,dz=\oint_C \frac{5}{z+3}dz$$

$\oint_C A\,dz=0$ **why?**

$$\oint_C \frac{5}{z+3}dz=2\pi i\,(5)=10\pi i=31.4i$$

Example IC65. Calculate $\oint_C \frac{z+12}{z^2+16}dz$ over C_1,C_2, C_3,



Solution: $\frac{z+12}{z^2+16}=\frac{A}{z+4i}+\frac{B}{z-4i}=\frac{A(z-4i)+B(z+4i)}{(z+4i)(z-4i)}$

$$\frac{z+12}{z^2+16}=\frac{(A+B)z-4iA+4iB}{z^2+16} ==> \begin{array}{c} A+B=1 \\ -4iA+4iB=12 \end{array}$$

B=1-A, -4i A + 4 i (1-A)=12 ==>-4iA -4iA=12-4i

$A=\frac{12-4i}{-8i}=0.5+1.5i$, B=1-A=0.5 -1.5i

Integration over C_1

$$\oint_{C1}\frac{z+12}{z^2+16}dz=\oint_{C1}\frac{A}{z+4i}dz+\oint_{C1}\frac{B}{z-4i}dz$$

C_1 encloses the point z=-4i.

$$\oint_{C1}\frac{A}{z+4i}dz=2\pi i\,A=2\pi i\,(0.5+1.5i)=-9.42+3.14i$$

$\oint_{C1}\frac{B}{z-4i}dz=0$ Because C_1 does not enclose z=4i

Thus

$$\oint_{C1}\frac{A}{z+4i}dz+\oint_{C1}\frac{B}{z-4i}dz=-9.42+3.14i+0=-9.42+3.14i$$

Result

$$\oint_{C1}\frac{z+12}{z^2+16}dz=-9.42+3.14i$$

Integration over C_2

$$\oint_{C2}\frac{z+12}{z^2+16}dz=\oint_{C2}\frac{A}{z+4i}dz+\oint_{C2}\frac{B}{z-4i}dz$$

$\oint_{C2}\frac{A}{z+4i}dz=0$ because C_2 does not enclose z= -4i

$$\oint_{C2}\frac{B}{z-4i}dz=2\pi i\,B=2\pi i\,(0.5-1.5i)=9.42+3.14i$$

Result

$$\oint_{C2}\frac{z+12}{z^2+16}dz=9.42+3.14i$$

Integration over C_3

$$\oint_{C3}\frac{z+12}{z^2+16}dz=\oint_{C3}\frac{A}{z+4i}dz+\oint_{C3}\frac{B}{z-4i}dz$$

C_3 encloses the point z=-4i. and z=4i.

$$\oint_{C3}\frac{A}{z+4i}dz=2\pi i\,A=2\pi i\,(0.5+1.5i)=-9.42+3.14i$$

$$\oint_{C3}\frac{B}{z-4i}dz=2\pi i\,B=2\pi i\,(0.5-1.5i)=9.42+3.14i$$

Thus

$$\oint_{C3}\frac{A}{z+4i}dz+\oint_{C3}\frac{B}{z-4i}dz=-9.42+3.14i+9.42+3.14i=6.28i$$

Result

$$\oint_{C3}\frac{z+12}{z^2+16}dz=6.28i$$