Cauchy's integral Formula for Derivatives of an analytic Function.

$$\oint_C \frac{f(z)}{(z-z_0)^2}dz = f'(z_0)\ 2\pi i$$

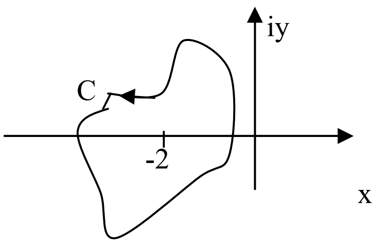
$$\oint_C \frac{f(z)}{(z-z_0)^3}dz = f''(z_0)\frac{2\pi i}{2!}$$

$$\oint_C \frac{f(z)}{(z-z_0)^4}dz = f'''(z_0)\frac{2\pi i}{3!}$$

In general

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = f^{(n)}(z_0)\frac{2\pi i}{n!}$$

Example A322 Calculate $\oint_C \frac{z^2+6}{(z+2)^2}dz$ over a closed curve that encloses the point z=-2.



Solution Here $f(z)=z^2+6$, and $z_0=-2$, → $f'(z)=2z$ and

$f'(z_0)=2z_0=2(-2)=-4$

$$\oint_C \frac{f(z)}{(z-z_0)^2}dz = 2\pi i\ f'(z_0)$$

$$\oint_C \frac{z^2+6}{(z-(-2))^2}dz = 2\pi i(-4) = -25.12i$$

Example A323 Calculate $\oint_C \frac{3z^2+6}{(z+2)^3}dz$ over a closed curve that encloses the point z=-2.

Solution Here $f(z)=3z^2+6$, and $z_0=-2$, → $f'(z)=6z$ and $f''(z)=6$ $f''(z_0)=6$

$$\oint_C \frac{f(z)}{(z-z_0)^3}dz = f''(z_0)\frac{2\pi i}{2} = 6\pi i$$

Example A324 Calculate $\oint_C \frac{3z^2+6}{(z+2)^4}dz$ over a closed curve that encloses the point z=-2.

Solution Here $f(z)=3z^2+6$, and $z_0=-2$, → $f'(z)=6z$ and $f''(z)=6$, $f'''(z)=0$, $f'''(z_0)=0$

$$\oint_C \frac{f(z)}{(z-z_0)^4}dz = f'''(z_0)\frac{2\pi i}{3!} = 0\ \frac{2\pi i}{3!} = 0$$

Example A325 Calculate $\oint_C \frac{3z^2+6}{(z+2)^5}dz$ over a closed curve that encloses the point z=-2. **The answer is zero. WHY?**

Example A326 Calculate $\oint_C \frac{1}{(z+2)^2}dz$ over a closed curve that encloses the point z=-2. **The answer is zero WHY?**

Example A327 Calculate $\oint_C \frac{z^3+11}{(z+2)^2(z+3)^2}dz$ over a closed curve that encloses the points z=-2 and z=-3.

$$\frac{z^3+1}{(z+2)^2(z+3)^2} = \frac{A}{(z+2)} + \frac{B}{(z+2)^2} + \frac{C}{(z+3)} + \frac{D}{(z+3)^2}$$

$$= \frac{A(z+2)(z+3)^2 + B(z+3)^2 + C(z+2)^2(z+3) + D(z+2)^2}{(z+2)^2(z+3)^2}$$

$$= \frac{A(z+2)(z+3)^2 + B(z+3)^2 + C(z+2)^2(z+3) + D(z+2)^2}{(z+2)^2(z+3)^2}$$

$$= \frac{(A+C)z^3 + (8A+B+7C+D)z^2 + (21A+6B+16C+4D)z}{(z+2)^2(z+3)^2}$$

$\underline{18A+9B+12C+4D}$

...

(A+C)=1, 8A+B+7C+D=0, 21A+6B+16C+4D=0, 18A+9B+12C+4D=11

A=6, B=3, C=-5, D=-16

$$\oint_C \frac{z^3+11}{(z+2)^2(z+3)^2}dz = \oint_C \frac{6}{(z+2)}dz + \oint_C \frac{3}{(z+2)^2}dz + $$
$$+ \oint_C \frac{-5}{(z+3)}dz + \oint_C \frac{-16}{(z+3)^2}dz$$

From Problem A326

$$\oint_C \frac{3}{(z+2)^2}dz = 0, \quad and \quad \oint_C \frac{-16}{(z+3)^2}dz = 0$$

$$\oint_C \frac{6}{(z+2)}dz = 6\ \ 2\pi i \qquad \oint_C \frac{-5}{(z+3)}dz = -5\ \ 2\pi i$$

Result: $\oint_C \frac{z^3+11}{(z+2)^2(z+3)^2}dz$ =12 πi -10 πi=2 πi=6.28i

$$\frac{15z^3+1}{(z+2)^3(z-1)^2(z+10)}=\frac{A_1}{(z+2)}+\frac{A_2}{(z+2)^2}+\frac{A_3}{(z+2)^3}+\frac{B_1}{(z-1)}+\frac{B_2}{(z-1)^2}+\frac{C}{(z+10)}$$

$$C=(z+10)F(z)\Big|_{z=-10}=\frac{15z^3+1}{(z+2)^3(z-1)^2}\Bigg|_{z=-10}$$

$$=\frac{15(-10)^3+1}{(-10+2)^3(-10-1)^2}=0.2421$$

$$B_2=(z-1)^2F(z)\Big|_{z=1}=\frac{15z^3+1}{(z+2)^3(z+10)}\Bigg|_{z=1}$$

$$=\frac{15(1)^3+1}{(1+2)^3(1+10)}=0.0539$$

$$B_1=\frac{d}{dz}\left[(z-1)^2F(z)\right]\Big|_{z=1}=\frac{d}{dz}\left[\frac{15z^3+1}{(z+2)^3(z+10)}\right]\Bigg|_{z=1}$$

$p=15z^3+1$ $\quad$ $p'=45z^2$

$q=(z+2)^3(z+10)$ $\quad$ $q'=3(z+2)^2(z+10)+(z+2)^3\,1$

$q'=4z^3+48z^2+144z+128$

$$\left(\frac{p}{q}\right)'=\left(\frac{p'q-pq'}{q^2}\right)$$

$$=\frac{-15z^6+1080z^4+3836z^3+3552z^2-144z-128}{(z+2)^6\,(z+10)^2}$$

$$\frac{-15z^6+1080z^4+3836z^3+3552z^2-144z-128}{(z+2)^6\,(z+10)^2}\Bigg|_{z=1}$$

$$\frac{-15\,1^6+1080\,1^4+3836\,1^3+3552\,1^2-144\,1-128}{(1+2)^6\,(1+10)^2}=0.092$$

$B_1=0.092$

$$A_3=(z+2)^3F(z)\Big|_{z=-2}=\frac{15z^3+1}{(z-1)^2(z+10)}\Bigg|_{z=-2}$$

$$=\frac{15(-2)^3+1}{(-2-1)^2(-2+10)}=-1.6528$$

$$A_2=\frac{d}{dz}\left[(z+2)^3F(z)\right]\Big|_{z=-2}=\frac{d}{dz}\left[\frac{15z^3+1}{(z-1)^2(z+10)}\right]\Bigg|_{z=-2}$$

$A_2=1.6047$

$$A_1=\frac{d^2}{dz^2}\left[(z+2)^3F(z)\right]\Big|_{z=-2}=\frac{d^2}{dz^2}\left[\frac{15z^3+1}{(z-1)^2(z+10)}\right]\Bigg|_{z=-2}$$

$A_1=-0.3349$

$$\frac{15z^3+1}{(z+2)^3(z-1)^2(z+10)}=\frac{-0.334}{(z+2)}+\frac{1.6}{(z+2)^2}+\frac{-1.65}{(z+2)^3}+\frac{0.092}{(z-1)}+\frac{0.053}{(z-1)^2}+\frac{0.242}{(z+10)}$$
