Taylor series

$$f(z)=f(z_0)+f'(z_0)(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2$$
$$+\frac{f'''(z_0)}{3!}(z-z_0)^3+....$$

Maclaurin Series (Taylor series for z_0=0)

$$f(z)=f(0)+f'(0)z+\frac{f''(0)}{2!}z^2+\frac{f'''(0)}{3!}z^3+....$$

AE-441 Calculate Taylor series of *f(z)=sin(z)*
Calculate sin(0.5), sin(π)

Solution

f(z)=sin(z), f'(z)=cos(z), f''(z)=-sin(z), f'''(z)=-cos(z),
for z=0;
f(0)=0, *f'*(0)= 1, *f''*(0)=0, *f'''*(0)=-1, $f^{(iv)}$ (0)=0, $f^{(v)}$ (0)=1,

$$\sin(z)=0+1z+\frac{0}{2!}z^2+\frac{-1}{3!}z^3+\frac{0}{4!}z^4+\frac{1}{5!}z^5+\frac{0}{6!}z^6....$$

$$\sin(z)=z-\frac{z^3}{6}+\frac{z^5}{120}-\frac{z^7}{5040}+\frac{z^9}{362880}-\frac{z^{11}}{39916800}+\cdots\cdots$$

set z=0.5

$$\sin(0.5)=0.5-\frac{0.5^3}{6}+\frac{0.5^5}{120}-\frac{0.5^7}{5040}+\frac{0.5^9}{362880}-\frac{0.5^{11}}{39916800}+\cdots\cdots$$

Iif we take the first term only sin(0.5)=0.5;

If we take the first two sin(0.5)=0.5 $\frac{0.5^3}{6}$ =0.47916;

The table shows sin(0.5) for higher terms

Number of Terms	Sin(0.5)	
1	0.5	
2	0.47916	
3	0.479427	
10	0.47942553860420	
25	0.47942553860420	

AE-443 Calculate sin(π)

$$\sin(\pi)=\pi-\frac{\pi^3}{6}+\frac{\pi^5}{120}-\frac{\pi^7}{5040}+\frac{\pi^9}{362880}-\frac{\pi^{11}}{39916800}+\cdots\cdots$$

The value of π is important here. Replace π=3.14159 and take 5 terms, we get sin(3.14159)=0.0004.

AE-444 Calculate sin(1+2i)

$$\sin(1+2i)=(1+2i)-\frac{(1+2i)^3}{6}+\frac{(1+2i)^5}{120}-\frac{(1+2i)^7}{5040}+\frac{(1+2i)^9}{362880}$$
$$-\frac{(1+2i)^{11}}{39916800}+\cdots\cdots = 3.1658+1.9596i$$

AE-445 Calculate sin(2i)

$$\sin(2i)=2i-\frac{(2i)^3}{6}+\frac{(2i)^5}{120}-\frac{(2i)^7}{5040}+\frac{(2i)^9}{362880}-\frac{(2i)^{11}}{39916800}+\cdots\cdots$$
=3.6269i

AE-446 Calculate Taylor series of *f(z)*= $\tan^{-1}$*(z)*
b)Calculate $\tan^{-1}$(1), **c)**Calculate π using $\tan^{-1}$(1).

Solution a)

f(z)= $\tan^{-1}$*(z)*, *f'(z)*= $\frac{1}{1+z^2}$,

$$\left(\frac{1}{1+z^2}\right)'=\frac{0-2z}{(1+z^2)^2}=f''(z)$$

similarly *f'''*(z)= $\frac{6z^2-2}{(1+z^2)^3}$, $f^{(iv)}$(z)= $\frac{-24z^3+24z}{(1+z^2)^4}$

for z=0
f(0)=0, *f'*(0)= 1, *f''*(0)=0, *f'''*(0)=-2, $f^{(4)}$ (0)=0, $f^{(5)}$ (0)=24, $f^{(6)}$ (0)=0, $f^{(7)}$ (0)=720, $f^{(8)}$ (0)=0, $f^{(9)}$ (0)=40320,

$$\tan^{-1}(z)=z-2\frac{z^3}{3!}+24\frac{z^5}{5!}-720\frac{z^7}{7!}+40320\frac{z^9}{9!}\cdots\cdots$$

$$\tan^{-1}(z)=z-\frac{z^3}{3}+\frac{z^5}{5}-\frac{z^7}{7}+\frac{z^9}{9}-\frac{z^{11}}{11}\cdots\cdots$$

b) replace z=1;

$$\tan^{-1}(1)=1-\frac{1^3}{3}+\frac{1^5}{5}-\frac{1^7}{7}+\frac{1^9}{9}-\frac{1^{11}}{11}\cdots\cdots= 0.7853981$$

c) since $\tan^{-1}(1) = 45^0 = \frac{\pi}{4}$ *radians*

$\frac{\pi}{4}$ = $\tan^{-1}$(1) =0.7853981 → π=4 x 0.7853981=3.1415924

AE 451 Calculate Taylor series of *f(z)*= $\sin^{-1}$*(z)*

f(z)= $\sin^{-1}$*(z)*, *f'(z)*= $\frac{1}{\sqrt{1-z^2}}$, *f''(z)*= $z\,(1-z^2)^{3/2}$

f'''(z)= $(2z^2+1)\,(1-z^2)^{5/2}$

set z=0 *f*(0)=0, *f'*(0)=1, *f''*(0)=0, *f'''*(0)=1, $f^{(4)}$(0)=0, $f^{(5)}$(0)=9, $f^{(6)}$(0)=0, $f^{(7)}$(0)=225, $f^{(8)}$(0)=0, $f^{(9)}$(0)=11025,

$$\sin^{-1}(z)=z+\frac{z^3}{3!}+225\frac{z^5}{5!}-11025\frac{z^7}{7!}+\cdots\cdots$$

AE 452 Calculate $\sin^{-1}$(3+4i)

$$\sin^{-1}(3+4i)=(3+4i)+\frac{(3+4i)^3}{3!}+225\frac{(3+4i)^5}{5!}-11025\frac{(3+4i)^7}{7!}+\cdot$$
=0.63+2.3i

AE 453 Calculate Taylor series of *f(z)*= ln(1+z)

f(z)= ln(1+z), *f'(z)*= $\frac{1}{1+z}$, *f''(z)* = $-(1+z)^{-2}$,

f'''(z)=2$(1+z)^{-3}$, $f^{(4)}$*(z)*=-6$(1+z)^{-4}$, $f^{(5)}$*(z)*=24$(1+z)^{-5}$,
Set z=0;
f(0)=0, *f'*(0)=1, *f''*(0)=-1, *f'''*(0)=2, $f^{(4)}$(0)=-6, $f^{(5)}$(0)=24, $f^{(6)}$(0)=-120, $f^{(7)}$(0)=720, $f^{(8)}$(0)=-5040, $f^{(9)}$(0)=40320,

$$\ln(1+z)=z-\frac{z^2}{2!}+2\frac{z^3}{3!}-6\frac{z^4}{4!}+24\frac{z^5}{5!}-120\frac{z^5}{6!}+720\frac{z^5}{7!}\cdots\cdots$$

$$\ln(1+z)=z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\frac{z^5}{5}-\frac{z^5}{6}+\frac{z^5}{7}\cdots\cdots$$

AE 454 Calculate ln(1.5)

$$\ln(1+0.5)=0.5-\frac{0.5^2}{2}+\frac{0.5^3}{3}-\frac{0.5^4}{4}+\frac{0.5^5}{5}-\frac{0.5^5}{6}\cdots\cdots=0.405$$