

Taylor Series for Function Approximation

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \dots \cong f(x_0) + f'(x_0)(x - x_0)$$

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

Example CT11- Obtain a linear approximation for the function $f(x)=e^{2x}$ around $x_0=3$.

Solution

$$f(x)=e^{2x} \quad f(x_0)=e^{2 \cdot 3}=e^6=403.4288$$

$$f'(x)=2e^{2x} \quad f'(x_0)=2e^{2 \cdot 3}=2e^6=806.8576$$

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \\ = 403.43 + 806.85(x-3) = 806.85x - 2017.143$$

$$e^{2x} \approx 806.85x - 2017.5 \quad (\text{around } x=3)$$

X	$806.85x - 2017.143$	e^{2x}	Error
2.9	322.5	330.29	7.7
2.95	362.9	365	2.12
3	403.2	403.2	0.0
3.05	443.5	445.8	2.26
3.1	483.9	492.7	8.81

$$\text{Error} = |e^{2x} - (806.85x - 2017.5)|$$

Example CT12- Obtain a linear approximation for the function $f(x)=e^{2x}$ around $x_0=0.1$.

Solution

$$f(x)=e^{2x} \quad f(x_0)=e^{2 \cdot 0.1}=e^{0.2}=1.221 \\ f'(x)=2e^{2x} \quad f'(x_0)=2e^{2 \cdot 0.1}=2e^{0.2}=2.442$$

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \\ = 1.221 + 2.442(x-0.1) = 2.442x + 0.977$$

$$e^{2x} \approx 2.442x + 0.977 \quad (\text{around } x=0.1)$$

Example CT13- Obtain a linear approximation for the function $f(x)=\sin(x)$ around $x_0=0.8$.

Solution

$$f(x)=\sin(x) \quad f(x_0)=\sin(0.8)=0.71735 \\ f'(x)=\cos(x) \quad f'(x_0)=\cos(0.8)=0.69670$$

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \\ = 0.71735 + 0.69635(x-0.8) = 0.69635x + 0.16$$

$$\sin(x) \approx 0.69635x + 0.16 \quad (\text{around } x=0.8)$$

X	$0.69635x + 0.16$	$\sin(x)$	Error
0.7	0.647	0.644	0.0032
0.8	0.717	0.717	0.000
0.9	0.786	0.783	0.0033

Example CT14- Obtain a linear approximation for the function $f(x)=\sin(x)$ around $x_0=0$.

Solution

$$f(x)=\sin(x) \quad f(x_0)=\sin(0)=0. \\ f'(x)=\cos(x) \quad f'(x_0)=\cos(0)=1$$

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \\ = 0 + 1(x-0) = x \\ \sin(x) \approx x \quad (\text{around } x=0)$$

$\sin(x)$	$\sin(x)$	Error
-0.2	0.1986	0.0013
-0.1	0.0998	0.00016
0	0	0
0.1	0.0998	0.00016
0.2	0.1986	0.0013
0.5	0.47942	0.0205
1	0.8414	0.158

Example CT15- Obtain a linear approximation for the function $f(x)=\cos(x)$ around $x_0=0$.

Solution

$$f(x)=\cos(x) \quad f(x_0)=\cos(0)=1. \\ f'(x)=\sin(x) \quad f'(x_0)=\sin(0)=0$$

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \\ = 1 + 0(x-1) = 1 \\ \cos(x) \approx 1 \quad (\text{around } x=0)$$

Example CT16- Obtain a linear approximation for the function $f(x)=e^{-4x^2+2}$ around $x=0.1$.

Solution

$$f(x)=e^{-4x^2+2} \quad f(0.1)=7.09932$$

$$f'(x)=-8x e^{-4x^2+2} \quad f'(0.1)=-5.6794 \\ f(x) \approx f(x_0) + f'(x_0)(x - x_0) \\ = 7.09932 + (-5.6794)(x - 0.1) \\ = -5.6794x + 7.6672$$