Use of Residues: Partial Fraction Expansion.

$$f(z)=\frac{p(z)}{q(z)}=\frac{A}{z-a}+\frac{B}{z-b}+\frac{C}{z-c}.+.......$$

First residue Formula

$$A=\operatorname{Res}(z=a)=\lim_{z\to a}(z-a)\frac{p(z)}{q(z)}$$

$$B=\operatorname{Res}(z=b)=\lim_{z\to b}(z-b)\frac{p(z)}{q(z)}$$

$$C=\operatorname{Res}(z=c)=\lim_{z\to c}(z-c)\frac{p(z)}{q(z)}$$

Second residue Formula

$$A=\operatorname{Res}(z=a)=\lim_{z\to a}\frac{p(z)}{q'(z)}$$

$$B=\operatorname{Res}(z=b)=\lim_{z\to b}\frac{p(z)}{q'(z)}$$

$$C=\operatorname{Res}(z=c)=\lim_{z\to c}\frac{p(z)}{q'(z)}$$

Example AE-611

$$\frac{z+5}{z^3-5z^2-2z+24}=\frac{z+5}{(z-4)(z+2)(z-3)}=\frac{A}{z-4}+\frac{B}{z+2}+\frac{C}{z-3}$$

$$A=\lim_{z\to 4}(z-4)\frac{p(z)}{q(z)}=\lim_{z\to 4}(z-4)\frac{z+5}{(z-4)(z-3)(z+2)}$$

$$=\lim_{z\to 4}\frac{z+5}{(z-3)(z+2)}=\frac{4+5}{(4-3)(4+2)}=\frac{9}{6}=1.5$$

$$B=\lim_{z\to -2}(z-(-2))\frac{p(z)}{q(z)}=\lim_{z\to -2}(z+2)\frac{z+5}{(z-4)(z-3)(z+2)}$$

$$=\lim_{z\to -2}\frac{z+5}{(z-3)(z-4)}=\frac{-2+5}{(-2-3)(-2-4)}=\frac{3}{30}=0.1$$

$$C=\lim_{z\to 3}(z-3)\frac{p(z)}{q(z)}=\lim_{z\to 3}\frac{z+5}{(z-4)(z+2)}=\frac{3+5}{(3-4)(3+2)}=-1.6$$

Using the **Second residue Formula**

In our problem $p(z)=z+5$ $q(z)=z^3-5z^2-2z+24$ and $q'(z)=3z^2-10z-2$

$$A=\lim_{z\to 4}\frac{z+5}{3z^2-10z-2}=\frac{4+5}{3\ 4^2-10\ \ 4-2}=\frac{9}{6}=1.5$$

$$B=\lim_{z\to -2}\frac{z+5}{3z^2-10z-2}=\frac{-2+5}{3\ (-2)^2-10\ (-2)-2}=\frac{3}{30}=0.1$$

$$C=\lim_{z\to 3}\frac{z+5}{3z^2-10z-2}=\frac{3+5}{3\ (3)^2-10\ (3)-2}=\frac{8}{-5}=-1.6$$

Note we get the same result by **classical method**

$$\frac{z+5}{z^3-5z^2-2z+24}=\frac{A}{z-4}+\frac{B}{z+2}+\frac{C}{z-3}$$

$$=\frac{A(z+2)(z-3)+B(z-4)(z-3)+C(z-4)(z+2)}{(z-4)(z+2)(z-3)}$$

$$=\frac{A(z^2-z-6)+B(z^2-7z-12)+C(z^2-2z-8)}{(z-4)(z+2)(z-3)}$$

$$=\frac{z^2(A+B+C)+z(-7A-B-2C)+12A-6B-8C}{(z-4)(z+2)(z-3)}$$

$$z^2(A+B+C)+z(-7A-B-2C)+12A-6B-8C=z+5$$

$$A+B+C=0,\qquad -7A-B-2C=1,\qquad 12A-6B-8C=5$$

Solving for A,B,C we get A=1.5 B=0.1 C=-1.6

$$\frac{z+5}{z^3-5z^2-2z+24}=\frac{1.5}{z-4}+\frac{0.1}{z+2}+\frac{-1.6}{z-3}$$

Example AE-612

$$\frac{2z+12}{z^2+2z+2}=\frac{2z+12}{[z-(-1+i)][z-(-1-i)]}=\frac{2z+12}{(z+1-i)(z+1+i)}$$

$$=\frac{A}{(z+1-i)}+\frac{B}{(z+1+i)}$$

Using the **first residue formula**

$$A=\lim_{z\to -1+i}(z+1-i)\frac{2z+12}{(z+1-i)(z+1+i)}=\lim_{z\to -1+i}\frac{2z+12}{(z+1+i)}$$

$$=\frac{2(-1+i)+12}{((-1+i)+1+i)}=\frac{2i+10}{2i}=\frac{2i}{2i}+\frac{10}{2i}=1-5i$$

$$B=\lim_{z\to -1-i}(z+1+i)\frac{2z+12}{(z+1-i)(z+1+i)}=\lim_{z\to -1-i}\frac{2z+12}{(z+1-i)}=1+5i$$

Using the **second residue formula**

$p(z)=2z+12$, $q(z)=z^2+2z+2$, $q'(z)=2z+2$

$$A=\lim_{z\to -1+i}\frac{p(z)}{q'(z)}=\frac{2z+12}{2z+2}=\frac{2(-1+i)+12}{2(-1+i)+2}=1-5i$$

$$B=\lim_{z\to -1-i}\frac{p(z)}{q'(z)}=\frac{2z+12}{2z+2}=\frac{2(-1-i)+12}{2(-1-i)+2}=1+5i$$

Using **classical Method**

$$\frac{2z+12}{z^2+2z+2}=\frac{A}{(z+1-i)}+\frac{B}{(z+1+i)}=\frac{A(z+1+i)+B(z+1-i)}{(z+1-i)(z+1+i)}$$

(A+B)=2 A(1+i)+B(1-i)=12 solution is A=1-5i , B=1+5i

Result $\dfrac{2z+12}{z^2+2z+2}=\dfrac{1+5i}{(z+1-i)}+\dfrac{1-5i}{(z+1+i)}$