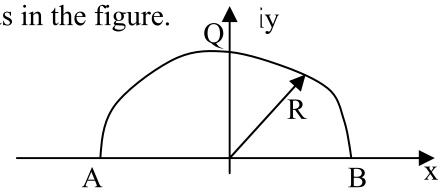
**Application of the residue theorem**
**Calculation of improper integrals**

**Example Problem AE-711**

**a)Calculate** $\oint_C \frac{1}{z^2+1}dz$ where C is the circle in upper left half plane as in the figure.




It is assumed that the radius R is very large.

b)Calculate the line integral $\oint_{BQA} \frac{1}{z^2+1}dz$ if R is infinity

c)Calculate the line integral $\int_{-\infty}^{+\infty} \frac{1}{x^2+1}dx$

**Solution : a)**The function $f(z)=\frac{1}{z^2+1}$ has two singular points $z=i$ and $z=-i$ . The point $z=i$ is inside the closed curve ABQA. The residue at $z=i$ is.

$$\operatorname{Res}(z=i)=\lim_{z\to i}(z-i)\frac{1}{(z-i)(z+i)}=\frac{1}{2i}=-0.5i$$

or $\operatorname{Res}(z=i)=\lim_{z\to i}\frac{1}{(z^2+1)'}=\lim_{z\to i}\frac{1}{2z}=\frac{1}{2i}=-0.5i$

Thus the integration over the closed curve ABQB is

$$\oint_{ABQA}\frac{1}{z^2+1}dz=2\pi i(-0.5i)=\pi$$

**b)**To calculate the line integral over the curve BQA replace $z=R\,e^{i\theta}$. R=constant. And $dz=R\,i\,e^{i\theta}\,d\theta$

$$\oint_{BQA}\frac{1}{z^2+1}dz=\int_{\theta=0}^{\theta=\pi}\frac{1}{(Re^{i\theta})^2+1}Rie^{i\theta}d\theta=\int_{\theta=0}^{\theta=\pi}\frac{Rie^{i\theta}}{(Re^{i\theta})^2+1}d\theta$$

Calculate the limit as $R\to\infty$

$$\lim_{R\to\infty}\int_{\theta=0}^{\theta=\pi}\frac{Rie^{i\theta}}{(Re^{i\theta})^2+1}d\theta=\int_{\theta=0}^{\theta=\pi}\lim_{R\to\infty}\left(\frac{Rie^{i\theta}}{(Re^{i\theta})^2+1}\right)d\theta$$

But $\lim_{R\to\infty}\left(\frac{Rie^{i\theta}}{(Re^{i\theta})^2+1}\right)=0$, Thus

$$=\int_{\theta=0}^{\theta=\pi}0\,d\theta=0$$

Result: if $R=\infty$ $\quad\oint_{BQA}\frac{1}{z^2+1}dz=0$

**c)** $\oint_{ABQA}\frac{1}{z^2+1}dz=\oint_{AB}\frac{1}{z^2+1}dz+\oint_{BQA}\frac{1}{z^2+1}dz$

Replace the values

$$\pi=\oint_{AB}\frac{1}{z^2+1}dz+0,\quad\to\quad\oint_{AB}\frac{1}{z^2+1}dz=\pi$$

Integration over the line AB is a straight line integral
Set y=0, dy=0 we get
z=x+iy=x+i0=x, dz=dx+idy=dx

$$\oint_{AB}\frac{1}{z^2+1}dz=\int_{-\infty}^{\infty}\frac{1}{x^2+1}dx=\pi$$

**Example Problem AE-712 Calculate** $\int_{-\infty}^{+\infty}\frac{1}{x^4+1}dx$

**Solution:** The solution is similar to problem AE-711. Integration over the line BQA is zero. Thus

$$\int_{-\infty}^{+\infty}\frac{1}{x^4+1}dx=\oint_{ABQA}\frac{1}{z^4+1}dz=2\pi i\left(\begin{matrix}\text{Residues of f(z)}\\ \text{inside ABQA}\end{matrix}\right)$$

$$=2\pi i\left((-\frac{4}{\sqrt{2}}-i\frac{4}{\sqrt{2}})+(\frac{4}{\sqrt{2}}-i\frac{4}{\sqrt{2}})\right)=\frac{\pi}{\sqrt{2}}$$

$z^4+1=0$ has four poles and two poles are inside the closed curve ABQA. These two poles are in upper left half plane as in the figure of problem AE-741.

**Example Problem AE-713 Calculate** $\int_{-\infty}^{+\infty}\frac{e^{iqx}}{x^2+1}dx$

**Solution:** The solution similar to problem AE-711. Integration over BQA is zero. The function

$$f(z)=\frac{e^{iqz}}{z^2+1}$$

Has one residue inside the closed curve ABQA.

$$\operatorname{Res}(z=i)=\lim_{z\to i}\frac{e^{iqz}}{(z^2+1)'}=\lim_{z\to i}\frac{e^{iqz}}{2z}=\frac{e^{iqi}}{2i}=-0.5e^{-q}i$$

$$\oint_{ABQA}\frac{e^{iqz}}{z^2+1}=2\pi i(-0.5e^{-q}i)=e^{-q}\pi$$

Result

$$\int_{-\infty}^{+\infty}\frac{e^{iqx}}{x^2+1}dx=e^{-q}\pi$$

**Example Problem AE-714** Using the above results

**Calculate** $\int_{-\infty}^{+\infty}\frac{\cos qx}{x^2+1}dx$, and $\int_{-\infty}^{+\infty}\frac{\sin qx}{x^2+1}dx$

**Solution:** $e^{iqx}=\cos(qx)+i\sin(qx)$

$$\int_{-\infty}^{+\infty}\frac{e^{iqx}}{x^2+1}dx=\int_{-\infty}^{+\infty}\frac{\cos qx}{x^2+1}dx+i\int_{-\infty}^{+\infty}\frac{\sin qx}{x^2+1}dx=e^{-q}\pi+0\,i$$

Then

$$\int_{-\infty}^{+\infty}\frac{\cos qx}{x^2+1}dx=e^{-q}\pi\quad\text{and}\quad\int_{-\infty}^{+\infty}\frac{\sin qx}{x^2+1}dx=0$$