MATRISIN OZDEGERLERI (EIGENVALUES)

Bir A matrisi icin
AX= λX olacak sekilde bir λ sayisi ve X vektoru bulunabiliyorsa bu λ sayisina A matrisinin ozdegeri X vektorune A matrisinin ozvektoru denir.

Ornek, A= $\begin{bmatrix} 12 & -3 \\ 8 & 2 \end{bmatrix}$, λ=6, X= $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 12 & -3 \\ 8 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}=\begin{bmatrix} 12x1-3x2 \\ 8x1+2x2 \end{bmatrix}=\begin{bmatrix} 6 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} 12 & -3 \\ 8 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}=6\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

A X = λX

λ=6 A matrisinin bir ozdegeridir. X= $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ vektoru A matrisinin bir ozvektorudur.

OZDEGERIN HESAPLANMASI
AX= λX → AX - λX =0, → (A – λI)X =0
Lineer cebir teorisinden bilindigi gibi (A – λI)X =0 denkleminin sifirdan farkli cozumu olabilmesi icin det(A – λI) =0 olmalidir.

x+2y=0
2x+4y=0

denkleminin (x=0, y=0) dan baska cozumu vardir. mesela (x=2, y=1) bir cozumdur. (x=10, y=5 de bir cozumdir.)

x+2y=0
x – y = 0

denkleminin sadece (x=0, y=0) cozumu vardir.
Lineer cebir teorisinde bu islem
matrisin satirlari lineer bagimli ise cozum yoktur
seklinde belirtilir.
Bir kare matrisin satirlari lineer bagimli ise
matrisin sutunlari da lineer bagimlidir.
matrisin determinanti sifirdir.

-------------------- -------------------

(A – λI)X =0 denkleminin sifirdan farkli cozumu olmasi icin det(A – λI)=0olmalidir.

-- ---

det(A – λI)=0 sartini saglayan λ degerleri A matrisinin ozdegerleridir.

Ornek61) A= $\begin{bmatrix} 12 & -3 \\ 8 & 2 \end{bmatrix}$ matrisinin ozdegerlerini ve ozvektorlerini bulun.

Cozum:

$$A- \lambda I = \begin{bmatrix} 12 & -3 \\ 8 & 2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 12-\lambda & -3 \\ 8 & 2-\lambda \end{bmatrix}$$

$$\det\begin{bmatrix} 12-\lambda & -3 \\ 8 & 2-\lambda \end{bmatrix} = (12-\lambda)(2-\lambda)-(8)(-3)$$

$=24- 12\lambda - 2\lambda + \lambda^2+24$
$= \lambda^2 - 14\lambda + 48$

det(A- λ I)=0 → $\lambda^2 - 14\lambda + 48=0$, $\lambda_1=6$, $\lambda_2=8$,

Ozdegerleri bulduk simdi de ozvektorl;eri bulalim.

(A- λ I)X=0, X= $\begin{bmatrix} a \\ b \end{bmatrix}$ diyelim ve a,b degerleini hesaplamaya calisalim.

$$\begin{bmatrix} 12-\lambda & -3 \\ 8 & 2-\lambda \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

λ=6 yerlestirelim.

$$\begin{bmatrix} 12-6 & -3 \\ 8 & 2-6 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 6 & -3 \\ 8 & -4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Iki bilinmeyenli iki denklem, matrisin satirlari lineer bagimli ve sonsuz cozum var. Bu cozumlerden herhangibirisi bizim icin ozvektordur.

6a-3b=0, → b=2a,

a=1, verelim → b=2 → X= $\begin{bmatrix} a \\ b \end{bmatrix}=\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

a=10, verelim → b=20 → X= $\begin{bmatrix} a \\ b \end{bmatrix}=\begin{bmatrix} 10 \\ 20 \end{bmatrix}$

a=7, verelim → b=14 → X= $\begin{bmatrix} a \\ b \end{bmatrix}=\begin{bmatrix} 7 \\ 14 \end{bmatrix}$

Hesaplanan butun vektorler birbirine lineer bagimlidir.

$$7\begin{bmatrix} 1 \\ 2 \end{bmatrix}=\begin{bmatrix} 7 \\ 14 \end{bmatrix}, \qquad 10\begin{bmatrix} 1 \\ 2 \end{bmatrix}=\begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

Dolayisiyla λ=6 icin tek bagimsiz bir ozvektor vardir. Yukarida hesaplanan ozvektorlerden herhangibirisi ozvektor olarak alinabilir.

--------------------- ----------------------

λ=8 yerlestirelim.

$$\begin{bmatrix} 12-8 & -3 \\ 8 & 2-8 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Yukaridakine benzer sekilde

$4c-3d=0, \rightarrow d=(3/4)c$, ve $X=\begin{bmatrix} c \\ d \end{bmatrix}=\begin{bmatrix} 1 \\ 3/4 \end{bmatrix}$

bulunur. Burada c=1 degeri verilmistir. (Herhangibir deger verilebilir.)

Sonuc :

$A=\begin{bmatrix} 12 & -3 \\ 8 & 2 \end{bmatrix}$ matrisinin iki ozdegeri vardir.

$\lambda_1=6 \qquad \lambda_2=8$

$\lambda_1=6$ icin ozvektor $X_1=\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda_1=8$ icin ozvektor $X_1=\begin{bmatrix} 1 \\ 3/4 \end{bmatrix}$

Problem AE-722. Find the eigenvalues and eigenvectors of the folllowing matrix.

$$A=\begin{bmatrix} 4 & 2 \\ -1 & 6 \end{bmatrix}$$

Solution:

$$A-\lambda I=\begin{bmatrix} 4 & 2 \\ -1 & 6 \end{bmatrix}-\lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 4-\lambda & 2 \\ -1 & 6-\lambda \end{bmatrix}$$

Det $(A-\lambda I)= (4-\lambda)(6-\lambda)-(-2)= \lambda2-10\lambda+26$

The roots are $\lambda_1=5+i$, $\lambda_2=5-i$

Find the eigenvector belong to $\lambda_1=5+i$

$$\begin{bmatrix} 4-\lambda_1 & 2 \\ -1 & 6-\lambda_1 \end{bmatrix}=\begin{bmatrix} 4-(5+i) & 2 \\ -1 & 6-(5+i) \end{bmatrix}=\begin{bmatrix} -1-i & 2 \\ -1 & 1-i \end{bmatrix}$$

$$\begin{bmatrix} -1-i & 2 \\ -1 & 1-i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}\rightarrow \begin{matrix} (-1-i)x_1+2x_2=0 \\ -x_1+(1-i)x_2=0 \end{matrix}$$

Set $x_1=1$, We get $2x_2=1+i, \rightarrow x_2=\frac{2}{1+i}=1-i$

Thus the eigenvector belong to $\lambda_1=5+i$, is

$$X_1=\begin{bmatrix} 1 \\ 1-i \end{bmatrix}$$

Similarly the eigenvector belong to $\lambda_1=5-i$, is

$$X_1=\begin{bmatrix} 1 \\ 1+i \end{bmatrix}$$

Problem AE-721. Find the eigenvalues and eigenvectors of the folllowing matrix.

$$A=\begin{bmatrix} 4 & 1 & 0 \\ -3 & -2 & 0 \\ 3 & 1 & 5 \end{bmatrix}$$

Solution:

$$A-\lambda I=\begin{bmatrix} 4 & 1 & 0 \\ -3 & -2 & 0 \\ 3 & 1 & 5 \end{bmatrix}-\lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 4-\lambda & 1 & 0 \\ -3 & -2-\lambda & 0 \\ 3 & 1 & 5-\lambda \end{bmatrix}$$

$$\text{Det}(A-\lambda I)=(5-\lambda)\begin{vmatrix} 4-\lambda & 1 \\ -3 & -2-\lambda \end{vmatrix}$$

$= (5-\lambda)\left((4-\lambda)(-2-\lambda)+3\right)$

$=(5-\lambda)(-8-4\lambda+2\lambda+\lambda^2+3)= (5-\lambda)(\lambda^2-2\lambda-5)$

$= -\lambda^3 + 7\lambda^2 -5\lambda-25$

The roots are $\lambda_1=3.45$, $\lambda_2=-1.45$, $\lambda_3=5$

Thus the eigenvalues are **$\lambda_1=3.45$, $\lambda_2=-1.45$, $\lambda_3=5$**

Now we shall find the eigenvectors.

First we calculate eigenvector belong to $\lambda_1=3.45$.

Set $\lambda_1=3.45$ in $[A-\lambda I]X=0$ and calculate the vector X.

$$\begin{bmatrix} 4-\lambda_1 & 1 & 0 \\ -3 & -2-\lambda_1 & 0 \\ 3 & 1 & 5-\lambda_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4-3.45 & 1 & 0 \\ -3 & -2-3.45 & 0 \\ 3 & 1 & 5-3.45 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0.55 & 1 & 0 \\ -3 & -5.45 & 0 \\ 3 & 1 & 1.55 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\rightarrow \begin{matrix} 0.55\,x1+x2=0, \\ -3\,x1-5.45\,x2=0, \\ 3x1+x2+1.55x3=0, \end{matrix}$$

Set one variable free. Set $x_1=1$

$0.55\, x_1+ x_2 =0; \rightarrow 0.55\ 1 + x_2 =0; \rightarrow x_2 =-0.55$

$-3\, x_1-5.45\, x_2 =0; \rightarrow x_2 =-0.55$

$3x_1+ x_2+1.55\, x_3=0; \rightarrow 3\ 1+ (-0.55)+1.55x_3=0;$

$x_3=\frac{-2.55}{1.55}=-1.645$

the eigenvector belong to the eigenvalue $\lambda_1=3.45$is

$$X_1=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 1 \\ -0.55 \\ -1.65 \end{bmatrix}$$

Now we calculate the **eigenvector belong to $\lambda_2=-1.45$.** Set $\lambda=-1.45$ in the equation $[A-\lambda I]X=0$ and calculate the vector X.

$$\begin{bmatrix} 4+1.45 & 1 & 0 \\ -3 & -2+1.45 & 0 \\ 3 & 1 & 5+1.45 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$5.55\, x_1+ x_2 =0;$

$-3\, x_1-0.55\, x_2 =0;$

$3x_1+ x_2+6.55x_3=0;$

Set $x_1=1$, calculate x_2
$5.55 x_1+ x_2 =0$; $x_2=-5.55$;
$3x_1+ x_2+6.55x_3=0$; → 3 1+ (-5.55)+6.55x_3=0;
$x_3=0.39$

The eigenvector belong to the eigenvalue $\lambda_1=-1.45$ is

$$X_2=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\-5.55\\0.39\end{bmatrix}$$

Now we calculate the eigenvector belong to $\lambda_3=5$
Set $\lambda=5$ in the equation $[A-\lambda I]X=0$ and calculate the vector X.

$$\begin{bmatrix}4-5 & 1 & 0\\-3 & -2-5 & 0\\3 & 1 & 5-5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}-1 & 1 & 0\\-3 & -7 & 0\\3 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\rightarrow\begin{matrix}-x1+x2=0,\\-3x1-7x2=0,\\3x1+x2=0,\end{matrix}$$

The coefficient of x_3 is zero in all three equations.
Any value of x_3 satisfy the equations.
Now we want to find the value of x_1 and x_2.
Set $x_1=1$ and calculate x_2
$-x_1+x_2=0$; → $-1+x_2=0$, → $x_2=1$
However $x_1=1$, $x_2=1$ does not satisfy the second and third equation.
$-3x_1-0.55x_2=0$; → -3 1-0.55 1 =0; → -4.55=0
$3x_1+x_2=0$; →3+1=0 → 4=0
The conflict can only be solved if we set $x_1=0$ and $x_2=0$. Thus the eigenvector belong to $\lambda_3=5$ is

$$X_3=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\\text{free}\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

We can set free variable x_3 any value. Here we set $x_3=1$. Notice: If we set $x_3=0$ then we lead to the trivial solution [0 0 0]. That was not our goal.

Comments: if X is an eigenvector, then αX is also an eigenvector.
Thus we have found that

$$X_1=\begin{bmatrix}1\\-0.55\\-1.65\end{bmatrix},\quad X_2=\begin{bmatrix}1\\-5.55\\0.39\end{bmatrix},\quad X_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

are eigenvectors. So $10X_1$, $100X_2$, $7X_3$ are also eigenvectors.

$$X_1=\begin{bmatrix}10\\-5.5\\-16.5\end{bmatrix},\quad X_2=\begin{bmatrix}100\\-555\\39\end{bmatrix},\quad X_3=\begin{bmatrix}0\\0\\7\end{bmatrix}$$

OZDEGERLERI ve OZVEKTORLERI MATLAB ile bulma
MATLAB da
>> [q1,q2]=eig(A)
yazarsaniz q1: ozvektorleri, q2: ozdegerleri verir

Ornek81) A=$\begin{bmatrix}5 & 3\\5 & 5\end{bmatrix}$ matrisinin ozdeger ve ozvektorlerini bulun.
Cevap: >>[q1,q2]=eig([5 3; 3 5])

$$q1=\begin{bmatrix}-0.7071 & 0.7071\\0.7071 & 0.7071\end{bmatrix},\quad q2=\begin{bmatrix}2 & 0\\0 & 8\end{bmatrix}$$

Burada $\lambda_1=2$, $X_1=\begin{bmatrix}-0.7071\\0.7071\end{bmatrix}$

$\lambda_2=8$, $X_2=\begin{bmatrix}0.7071\\0.7071\end{bmatrix}$

Yada

$\lambda_1=2$, $X_1=\begin{bmatrix}-1\\1\end{bmatrix}$ ve $\lambda_2=8$, $X_2=\begin{bmatrix}1\\1\end{bmatrix}$

seklinde de olabilir.

Diferansiyel Denklem Sistemleri

$$\frac{dq}{dt} = 3q \quad \rightarrow \quad q(t) = C\,e^{3t}$$
$$\frac{dq}{dt} = -5q \quad \rightarrow \quad q(t) = C\,e^{-5t}$$
$$\frac{dq}{dt} = aq \quad \rightarrow \quad q(t) = C\,e^{at}$$

$$\begin{aligned}\frac{dq_1}{dt} &= aq_1 + bq_2 \\ \frac{dq_2}{dt} &= cq_1 + dq_2\end{aligned} \quad \rightarrow \quad \begin{bmatrix} \frac{dq_1}{dt} \\ \frac{dq_2}{dt} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

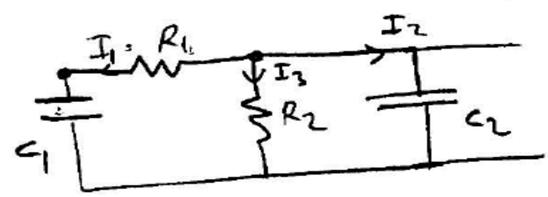
$q_1(t) = ?$ $\qquad$ $q_2(t) = ?$

En Genel Halde dif denklem

$$\begin{bmatrix} \frac{dq_1}{dt} \\ \frac{dq_2}{dt} \\ \dots \\ \frac{dq_n}{dt} \end{bmatrix} = \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ .. \\ .. \\ q_n \end{bmatrix}$$

Seklinde verilir. Burada A nxn boyutunda bir matrisdir. Bu tur denklemler Mekanik ve elektrik devrelerinde karsimiza cikar. Bu sekildeki denklemelere DURUM DENKLEMLERI (STATE SPACE EQUATION) denir. Durum denklemelerinin ozellileri butun turevli terimlerin denklemin sol tarafinda olacak sekilde duzenlenmesidir.

Ornek 511)



$$\begin{bmatrix} \frac{dV_{C1}}{dt} \\ \frac{dV_{C2}}{dt} \end{bmatrix} = \begin{bmatrix} \frac{1}{R_1C_1} & -\frac{1}{R_1C_1} \\ -\frac{R_2}{R_1C_2} & \frac{1}{C_2}\left(\frac{R_2}{R_1}+1\right) \end{bmatrix} \begin{bmatrix} V_{C1} \\ V_{C2} \end{bmatrix}$$

Ornek 512)

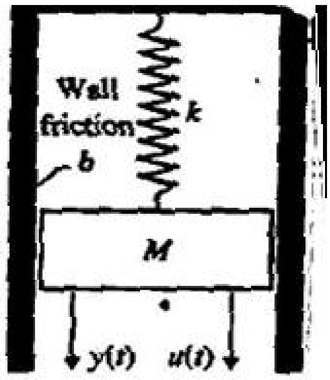
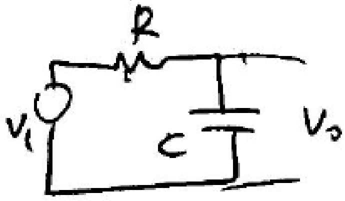


FIGURE 3.3
A spring-mass-damper system.

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

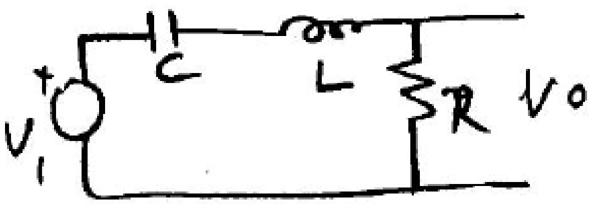
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Ornek 516)



$$\frac{dV_c}{dt} = -\frac{1}{RC} V_c + \frac{1}{RC} V_1$$

Ornek 521)



$$\frac{d}{dt}\begin{bmatrix} I_L \\ V_c \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} I_L \\ V_c \end{bmatrix} + \begin{bmatrix} \quad \end{bmatrix} V_1$$

Lineer Diferansiyel Denklem Sistemlerinin Homojen Cozumleri

En Genel Halde dif denklem

$$\begin{bmatrix} \frac{dq_1}{dt} \\ \frac{dq_2}{dt} \\ \cdots \\ \frac{dq_n}{dt} \end{bmatrix} = \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ .. \\ .. \\ q_n \end{bmatrix}$$

seklinde verildiginde cozum.

$$\begin{bmatrix} q_1(t) \\ q_2(t) \\ .. \\ .. \\ q_n \end{bmatrix} = C_1 \begin{bmatrix} \\ X_1 \\ \\ \end{bmatrix} e^{\lambda_1 t} + C_2 \begin{bmatrix} \\ X_2 \\ \\ \end{bmatrix} e^{\lambda_2 t} + ...+ C_n \begin{bmatrix} \\ X_n \\ \\ \end{bmatrix} e^{\lambda_n t}$$

seklindedir. Burada $\lambda_1,\lambda_2,\ldots \lambda_n$, A matrisinin ozdegerleri ve $X_1,X_2,\ldots X_n$, A matrisinin ozvektorleridir.

-------------------------- ------

$C_1,C_2,\ldots C_n$ denklemin baslangic kosullarina ait integral sabitleridir.

Example AE-782 Find the solution of the following differential equation.

$$\begin{bmatrix} \frac{dq_1}{dt} \\ \frac{dq_2}{dt} \\ \frac{dq_3}{dt} \\ \frac{dq_4}{dt} \end{bmatrix} = \begin{bmatrix} 3 & 7 & -12 & 7 \\ 4 & 3 & -30 & 21 \\ -0.5 & 0 & 3 & 0 \\ -1 & 0 & 7 & -1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_n \end{bmatrix}$$

Eigenvalues and eigenvectors of the matrix A is

λ_1=7.47, λ_2=4.03, λ_3=-3.15, λ_4=-0.35,

$$X_1 = \begin{bmatrix} -8.2 \\ -5.4 \\ 0.92 \\ 1.73 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 7 \\ 1.32 \\ -3.41 \\ -6.13 \end{bmatrix}, \quad X_3 = \begin{bmatrix} -7.2 \\ 6.76 \\ -0.59 \\ -1.44 \end{bmatrix}, X_4 = \begin{bmatrix} -9.48 \\ 2.77 \\ -1.42 \\ -0.66 \end{bmatrix}$$

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = C_1 \begin{bmatrix} -8.2 \\ -5.4 \\ 0.92 \\ 1.73 \end{bmatrix} e^{7.4t} + C_2 \begin{bmatrix} 7 \\ 1.32 \\ -3.41 \\ -6.13 \end{bmatrix} e^{4t} + C_3 \begin{bmatrix} -7.2 \\ 6.76 \\ -0.59 \\ -1.44 \end{bmatrix} e^{-3.1t} + C_4 \begin{bmatrix} -9.48 \\ 2.77 \\ -1.42 \\ -0.66 \end{bmatrix} e^{-0.35t}$$

or

$$q_1(t) = -C_1 8.2e^{7.4t} + C_2 7e^{4t} - C_3 7.2e^{-3.1t} - C_4 9.48e^{-0.35t}$$

$$q_2(t) = -C_1 5.4e^{7.4t} + C_2 1.3e^{4t} + C_3 6.7e^{-3.1t} + C_4 2.7e^{-0.35t}$$

q_3(t)=

q_4(t)=

Example AE-783 Find the solution of the the differential equation $\frac{dq}{dt} = 4q$ with the initial condition q(0)=2;

Solution: q(t)=Ce^{4t} replace t=0.

q(0)=Ce$^{4\,0}$

replace q(0)=2

2=Ce^{0} $\rightarrow$ C=2

Thus the solution is q(t)=2e^{4t}

Example AE-784 Find the solution of the the differential equation $\frac{dq}{dt} = 5q$ with the initial condition q(1)=3;

Solution: q(t)=Ce^{5t} replace t=1.

q(1)=Ce$^{5\,1}$

replace q(1)=3

3=Ce^{5} $\rightarrow$ C=3e^{-5}=3 0.0067= 0.0202

Thus the solution is q(t)=0.0202e^{5t}

Example AE-785 Find the solution of the following differential equation.

$$\begin{bmatrix} \frac{dp}{dt} \\ \frac{dq}{dt} \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}$$ with initial condition p(0)=60,

q(0)=20;

Solution λ_1=2, λ_2=8, $X_1=\begin{bmatrix} -7 \\ 7 \end{bmatrix}$, $X_2=\begin{bmatrix} -4.4 \\ 9 \end{bmatrix}$

p(t)=-7C$_1$e^{2t} - 4.4C$_2$e^{8t}

q(t)=7C$_1$e^{2t} - 9C$_2$e^{8t}

replace t=0

p(0)=-7C$_1$e$^{2\,0}$ - 4.4C$_2$e$^{8\,0}$

q(0)=7C$_1$e$^{2\,0}$ - 9C$_2$e$^{8\,0}$

60=-7C$_1$ - 4.4C$_2$

20=7C$_1$ - 9C$_2$

$C_1=-6.7 \quad C_2=-3$

Thus the required solution is

p(t)= -7 6.7 e^{2t} - 4.4(-3) e^{8t}=-46.9e^{2t} – 13.2 e^{8t}

q(t)=7 6.7e^{2t} - 9(-3) e^{8t}=46.9e^{2t} – 27 e^{8t}

Example AE-786 Find the solution of the following differential equation.

$$\begin{bmatrix} \frac{dp}{dt} \\ \frac{dq}{dt} \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -1 & 6 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}$$ with initial condition p(0)=1, q(0)=2;

Solution:

Eigenvalues and eigenvectors are

$\lambda_1=5+i$, $\lambda_2=5-i$, $X_1=\begin{bmatrix} 1 \\ 1-i \end{bmatrix}$, $X_2=\begin{bmatrix} 1 \\ 1+i \end{bmatrix}$

Thus the solution is

$$p(t) = C_1\, 1\, e^{(5+i)t} + C_2\, 1\, e^{(5-i)t}$$

$$q(t) = C_1\, (1-i)\, e^{(5+i)t} + (1+i)C_2\, e^{(5-i)t}$$

Now find the constant coefficients C_1 and C_2.

It is given that p(0)=1 and q(0)=2. Thus replace t=0 in the solution equations

$$p(0) = C_1\, 1\, e^{(5+i)0} + C_2\, 1\, e^{(5-i)0} \quad \text{or}$$

$$1=C_1+C_2$$

$$q(0) = C_1\, (1-i)\, e^{(5+i)0} + (1+i)C_2\, e^{(5-i)0} \quad \text{or}$$

$$2 = C_1\, (1-i) + (1+i)C_2$$

From the first equation $C_1=1-C_2$

Substitute this C_1 value into second equation

$$2 = (1-C_2)\,(1-i) + (1+i)C_2$$

$$2 = (1-i) - (1-i)C_2 + (1+i)C_2$$

2=(1-i)+2i C_2 $\rightarrow$ $C_2 = \dfrac{1+i}{2i} = 0.5 - 0.5i$

$C_1=1-C_2=1-(0.5-0.5i)=0.5+0.5i$

Thus the solutions is

$$p(t) = (0.5+0.5i)\, e^{(5+i)t} + (0.5-0.5i)\, 1\, e^{(5-i)t}$$

$$q(t) = (0.5+0.5i)\,(1-i)\, e^{(5+i)t} + (0.5-0.5i)(1+i)e^{(5-i)t}$$

Note

$$p(t) = (0.5+0.5i)\, e^{(5+i)t} + (0.5-0.5i)\, 1\, e^{(5-i)t}$$

$$= (0.5+0.5i)\, e^{5t} e^{it} + (0.5-0.5i)\, e^{5t} e^{-it}$$

$$= e^{5t}\left((0.5+0.5i)\, e^{it} + (0.5-0.5i)\, 1\, e^{-it}\right)$$

$$= e^{5t}\left(0.5e^{it} + 0.5e^{-it} + 0.5i\, e^{it} - 0.5i\, e^{-it}\right)$$

$$= e^{5t}\left(0.5(e^{it} + e^{-it}) + 0.5i\,(e^{it} - e^{-it})\right)$$

replace $\dfrac{e^{it} + e^{-it}}{2} = \cos t \qquad \dfrac{e^{it} - e^{-it}}{2i} = \sin t$

$$= e^{5t}\left(0.5(2\cos t) + 0.5i\,(2i\sin t)\right)$$

$$= e^{5t}\left(\cos t - \sin t\right)$$

In a similar manner

$$q(t) = (0.5+0.5i)\,(1-i)\, e^{(5+i)t} + (0.5-0.5i)(1+i)e^{(5-i)t}$$

$$= e^{(5+i)t} + e^{(5-i)t} = e^{5t}(e^{it} + e^{-it}) = 2e^{5t}\cos t$$

Result the solution is

$p(t)=e^{5t}\left(\cos t - \sin t\right) \qquad q(t)=2e^{5t}\cos t$

Example AE-786 Find the solution of the following differential equation.

$$\begin{bmatrix} \frac{dp}{dt} \\ \frac{dq}{dt} \\ \frac{dr}{dt} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3.1 \\ 3 & 7 & 2 \\ 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Solution: $\lambda_1=7$, $\lambda_1=1.5+3i$, $\lambda_2=1.5-3i$,

$X_1=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $X_2=\begin{bmatrix} -6.58-1.09i \\ 3.5+0.13i \\ 6.5i \end{bmatrix}$, $X_3=\begin{bmatrix} -6.58+1.09i \\ 3.5-0.13i \\ -6.5i \end{bmatrix}$,

$$p(t) = C_1 e^{7t} + e^{1.5t}(A\cos 3t + B\sin 3t)$$

$$q(t) = C_2 e^{7t} + e^{1.5t}(D\cos 3t + E\sin 3t)$$

$$r(t) = C_3 e^{7t} + e^{1.5t}(F\cos 3t + G\sin 3t)$$

Durum Denklemleri

Onceki bolumlerde bahsedildigi gibi durum denklemleri

$$\begin{bmatrix} \frac{dq_1}{dt} \\ \frac{dq_2}{dt} \\ \dots \\ \frac{dq_n}{dt} \end{bmatrix} = \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ .. \\ .. \\ q_n \end{bmatrix}$$

seklinde veriliyordu.

Yuksek mertebeden diferansiyel denklemler durum denklemi haline getirilebilir.

Ornek 56)

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 8y = f(t) \quad (1)$$

Dif denklemini durum denklemi haline getirin.

Cozum:

$$x_1 = y, \quad x_2 = \frac{dy}{dt}, \quad (2)$$

Tanimlarini yapalim bu durumda

$$x_2 = \frac{dx_1}{dt}, \quad \frac{d^2y}{dt^2} = \frac{dx_2}{dt} \quad (3)$$ olacagi aciktir.

Yukaridaki (1) denklemini bu tanimlara gore yazalim. .

$$\frac{dx_2}{dt} + 5x_2 + 8x_1 = f(t) \quad \text{veya}$$

$$\frac{dx_2}{dt} = -5x_2 - 8x_1 + f(t) \quad (4)$$

(2) de verilen x_2 tanimi ile (4) denklemini alt alta yazalim.

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = -5x_2 - 8x_1 + f(t) \text{ ss}$$

Matris halinde yazalim

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -8 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} f(t)$$

Ornek 57)

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x - 5U = 0$$

define $x_1 = x$

$$x_2 = \frac{dx_1}{dt} = \frac{dx}{dt}$$

then $\frac{d^2x}{dt^2} = \frac{dx_2}{dt}$

$$\frac{dx_2}{dt} + 3x_2 + 2x_1 - 5U = 0$$

or

$$\frac{dx_2}{dt} = -3x_2 - 2x_1 + 5U$$

$$\frac{dx_1}{dt} = x_2$$

$$\frac{d}{dt}\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \end{bmatrix} U$$

NOT: denklem asagidaki sekilde de duzenlenebilir.

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = -3x_2 - 2x_1 + 5U$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix} U$$

Ornek 61)

$$\frac{d^3x}{dt^3}+2\frac{d^2x}{dt^2}+8\frac{dx}{dt}+3x=u(t)$$

Define $x_1=x$

$$x_2=\frac{dx}{dt}=\frac{dx_1}{dt}$$

$$x_3=\frac{dx_2}{dt}=\frac{d^2x_1}{dt^2}=\frac{d^2x}{dt^2}$$

x_3 un turevini alirsak

$$\frac{d^3x}{dt^3}=\frac{d}{dt}\left(\frac{d^2x}{dt^2}\right)=\frac{dx_3}{dt}$$

Orijinal denklemler yeniden yazilirsa

$$\frac{dx_3}{dt}+2x_3+8x_2+3x_1=u(t)$$

$$\frac{dx_3}{dt}=-2x_3-8x_2-3x_1+u(t)$$

$$\frac{dx_2}{dt}=x_3$$

$$\frac{dx_1}{dt}=x_2$$

$$\frac{d}{dt}\begin{bmatrix}x_3\\x_2\\x_1\end{bmatrix}=\begin{bmatrix}-2&-8&-3\\1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}x_3\\x_2\\x_1\end{bmatrix}+\begin{bmatrix}1\\0\\0\end{bmatrix}u(t)$$

Ornek 63)

$$\frac{d^3x}{dt^3}+x=u(t)$$

$$x_1=x$$

$$x_2=\frac{dx}{dt}=\frac{dx_1}{dt}$$

$$x_3=\frac{dx_2}{dt}=\frac{d^2x_1}{dt^2}\Rightarrow\frac{dx_3}{dt}=\frac{d^3x_1}{dt^3}$$

$$\frac{dx_3}{dt}+x_1=u(t)$$

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\-1&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}u$$