EC-101

8.1 ◆ Introduction to the Natural Response of a Parallel *RLC* Circuit

Parallel RLC circuits

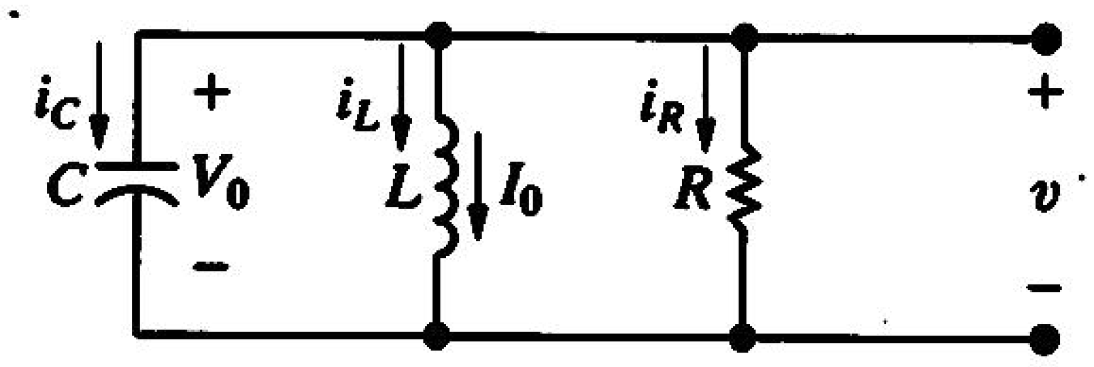


Figure 8.1 A circuit used to illustrate the natural response of a parallel *RLC* circuit.

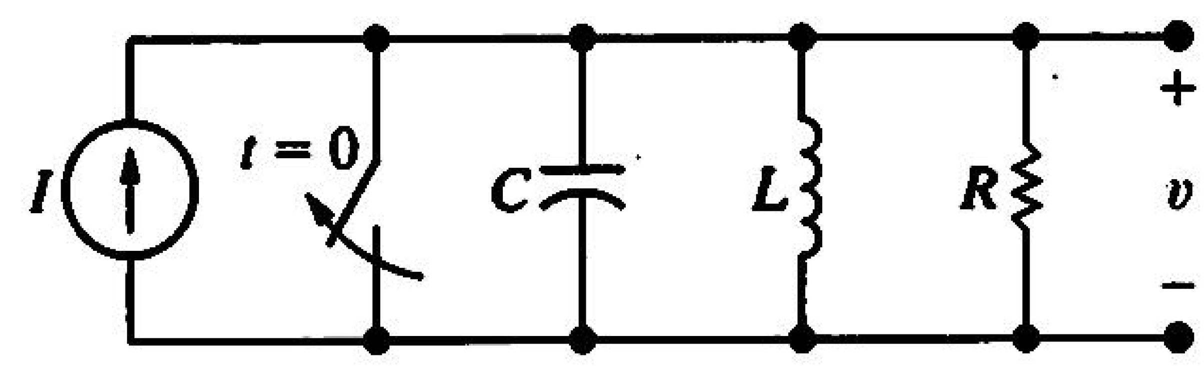


Figure 8.2 A circuit used to illustrate the step response of a parallel *RLC* circuit.

$$-I_o = I_c + I_L + I_R$$

$$\frac{dI_o}{dt} = \frac{dI_c}{dt} + \frac{dI_L}{dt} + \frac{dI_R}{dt}$$

$$I_o = \text{constant} \qquad \frac{dI_o}{dt} = 0$$

$$I_c = c\frac{dV_c}{dt} \Rightarrow \frac{dI_c}{dt} = c\frac{d^2V_c}{dt^2}$$

$$V_L = L\frac{dI_L}{dt} \Rightarrow \frac{dI_L}{dt} = \frac{V_L}{L}$$

$$I_R = \frac{V_R}{R} \Rightarrow \frac{dI_R}{dt} = \frac{1}{R}\frac{dV_R}{dt}$$

$$V_L = V_c = V_R = V$$

$$\frac{dI_o}{dt} = \frac{dI_c}{dt} + \frac{dI_L}{dt} + \frac{dI_R}{dt}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$0 \qquad c\frac{dv^2}{dt^2} + \frac{V}{L} + \frac{1}{R}\frac{dv}{dt}$$

$$\frac{dv^2}{dt} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

Second order diff. equation

The General Solution of the Second-Order Differential Equation

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$$

$s^2 + bs + c = 0$ roots are $s_{1,2} = \dfrac{-b \mp \sqrt{b^2-4c}}{2}$

$x(t) = d_1 e^{s_1 t} + d_2 e^{s_2 t}$ (d_1, d_2 constants)

Example: $\dfrac{d^2x}{dt^2} + 3\dfrac{dx}{dt} + 2x = 0$

$s^2 + 3s + 2 = 0$ $s_1 = \dfrac{-3 + \sqrt{3^2 - 4\times 2}}{2} = -1$

$s_2 = -2$

$$\boxed{x(t) = d_1 e^{-t} + d_2 e^{-2t}}$$

Example problem: Solve $\dfrac{d^2x}{dt^2} - 5\dfrac{dx}{dt} + 6x = 0$ $x(0) = 2$

$\left.\dfrac{dx}{dt}\right|_{t=0} =$

$s^2 - 5s + 6 = 0 \Rightarrow s_1 = 3$ $s_2 = 2$

$x(t) = d_1 e^{3t} + d_2 e^{2t}$

$\dfrac{dx}{dt} = 3d_1 e^{3t} + 2d_2 e^{2t}$

$x(0) = d_1 e^0 + d_2 e^0 =$ $\Rightarrow$ $d_1 + d_2 = 9$

$\left.\dfrac{dx}{dt}\right|_{t=0} = 3d_1 e^0 + 2d_2 e^0 =$ $\Rightarrow$ $3d_1 + 2d_2 = 22$

$\begin{pmatrix} d_1 + d_2 = 9 \\ 3d_1 + 2d_2 = 22 \end{pmatrix} \Rightarrow \begin{matrix} d_1 = 4 \\ d_2 = 5 \end{matrix}$

$x(t) = 4e^{3t} + 5e^{2t}$

Example: solve $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} = 0$ $\quad x(0) = 10 \quad \left.\frac{dx}{dt}\right|_{t=0} = 20$

Solution $\quad s^2 + 4s = 0 \quad s_1 = -4, \; s_2 = 0$

$$x(t) = d_1 e^{-4t} + d_2 e^{0t} = d_1 e^{-4t} + d_2$$

$$\frac{dx}{dt} = -4 d_1 e^{-4t}$$

$$x(0) = d_1 + d_2 = 10$$

$$\left.\frac{dx}{dt}\right|_{t=0} = -4 d_1 = 20 \Rightarrow d_1 = -5$$

$$d_2 = 15$$

$$x(t) = -5 e^{-4t} + d_2$$

Example $\quad \frac{d^2x}{dt^2} - 9x = 0 \quad x(0) = \quad x'(0) = \left.\frac{dx}{dt}\right|_{t=0} =$

$$s^2 - 9 = 0 \quad s_1 = 3 \quad s_2 = -3$$

$$x(t) = d_1 e^{3t} + d_2 e^{-3t}$$

$$\frac{dx}{dt} = 3 d_1 e^{3t} - 3 d_2 e^{-3t}$$

$$\left.\begin{aligned} x(0) &= d_1 + d_2 = 6 \\ x'(0) &= 3d_1 - 3d_2 = 6 \end{aligned}\right) \Rightarrow \begin{aligned} d_1 &= 4 \\ d_2 &= 2 \end{aligned}$$

$$x(t) = 4 e^{3t} + 2 e^{-3t}$$

Complex roots

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 13x = 0$$

$$s^2 + 4s + 13 = 0 \qquad s_1 = \frac{-4+\sqrt{16-4\times 13}}{2} = \frac{-4+6i}{2} = -2+3j$$

$$s_2 = \frac{-4-\sqrt{16-4\times 13}}{2} = -2-3j$$

$$x(t) = d_1 e^{(-2+3j)t} + d_2 e^{(-2-3j)t}$$

Or

$$x(t) = e^{-2t}(A\cos 3t + B\sin 3t)$$

Example: $\frac{d^2x}{dt^2} - 6\frac{dx}{dt} + 13x = 0 \qquad x(0) = 5 \qquad \left.\frac{dx}{dt}\right|_{t=0} = 19$

Solution: $s^2 - 6s + 13 = 0 \Rightarrow s_1 = 3+2j \qquad s_2 = 3-2j$

$$x(t) = e^{3t}(A\cos 2t + B\sin 2t)$$

$$x(0) = e^0(A\cos 0 + B\sin 0) = A = 5$$

$$\frac{dx}{dt} = 3e^{3t}(A\cos 2t + B\sin 2t) + e^{3t}(-2A\sin 2t + 2B\cos 2t)$$

$$\left.\frac{dx}{dt}\right|_{t=0} = 3e^0(A+0) + e^0(-0+2B) = 3A + 2B = 19$$

$$3\times 5 + 2B = 19 \Rightarrow B = 2$$

$$x(t) = e^{3t}(5\cos 2t + 2\sin 2t)$$

Example: $\dfrac{d^2x}{dt^2} + 9x = 0 \qquad x(0) = 5 \qquad x'(0) = 18$

Solution

$$s^2 + 9 = 0 \Rightarrow s_1 = 3j \qquad s_2 = -3j$$

$$x(t) = A \cos 3t + B \sin 3t$$

$$\frac{dx}{dt} = -3A \sin 3t + 3B \cos 3t$$

$$x(0) = A = 5$$

$$\left.\frac{dx}{dt}\right|_{t=0} = x'(0) = 3B = 18 \Rightarrow B = 6$$

$$x(t) = 5 \cos 3t + 6 \sin 3t$$

Note

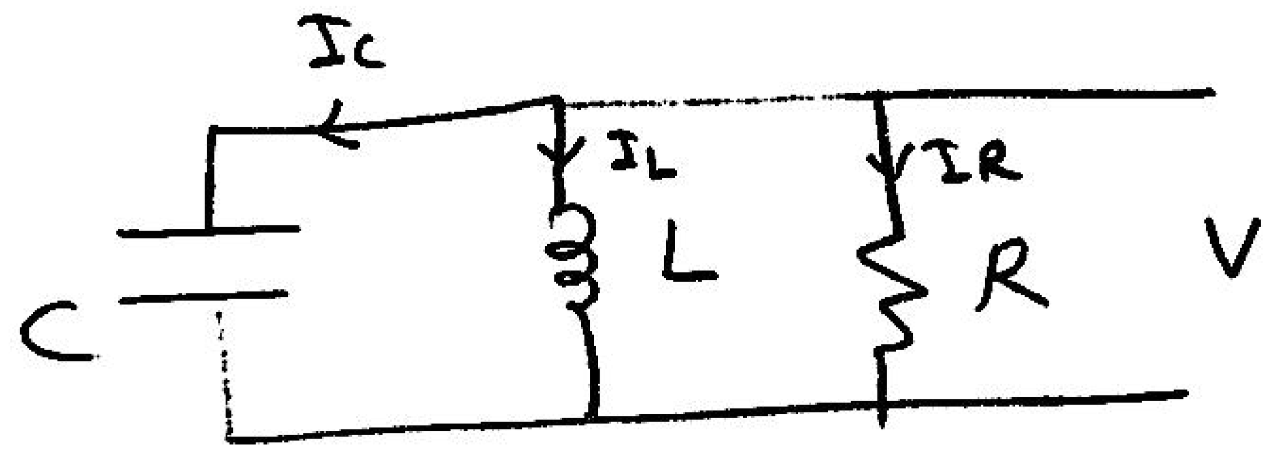
$$as^2 + bs + c = 0$$

$$\Delta = b^2 - 4ac$$

$\Delta > 0$ two real roots

$\Delta = 0$ equal roots $(s_1 = s_2)$

$\Delta < 0$ complex roots



$$\frac{dV^2}{dt} + \frac{1}{RC}\frac{dV}{dt} + \frac{1}{LC}V = 0$$

$$\Delta = \left(\frac{1}{RC}\right)^2 - \frac{4}{LC} \qquad S_{1,2} = \frac{-\frac{1}{RC} \mp \sqrt{\Delta}}{2} = -\frac{1}{2RC} \mp \sqrt{\frac{\Delta}{4}}$$

$$\alpha = \frac{1}{2RC} \qquad w_0 = \frac{1}{\sqrt{LC}} \rightarrow \text{resonant frequency}$$

$$S_1 = -\alpha + \sqrt{\alpha^2 - w_0^2}$$

$$S_2 = -\alpha - \sqrt{\alpha^2 - w_0^2}$$

$\Delta > 0 \Rightarrow \alpha^2 > w_0^2 \Rightarrow$ two real roots, Over damped

$\Delta < 0 \Rightarrow \alpha^2 < w_0^2 \Rightarrow$ Complex roots, Under damped

$\Delta = 0 \Rightarrow \alpha^2 = w_0^2 \Rightarrow$ equal roots, critically damped

8.1

a) Find the roots of the characteristic equation that governs the transient behavior of the voltage shown in Fig. 8.5 if $R = 200\ \Omega$, $L = 50$ mH, and $C = 0.2\ \mu F$.

b) Will the response be overdamped, underdamped, or critically damped?

c) Repeat (a) and (b) for $R = 312.5\ \Omega$.

d) What value of R causes the response to be critically damped?

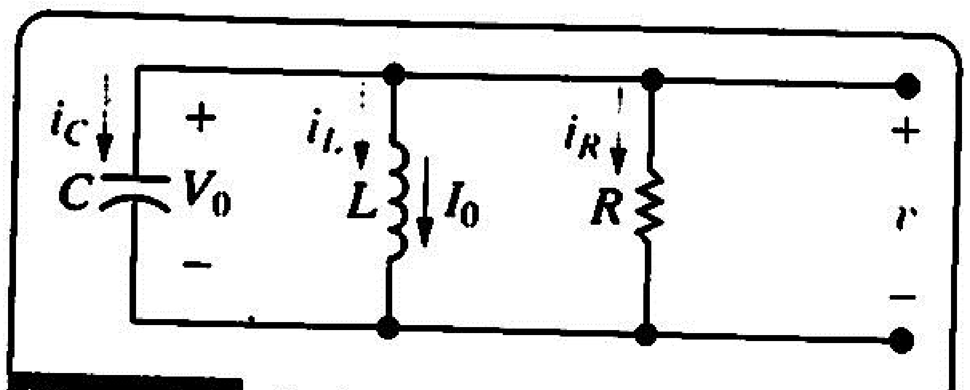


Figure 8.5 A circuit used to illustrate the natural response of a parallel *RLC* circuit.

solution $$\frac{d^2V}{dt} + \frac{1}{RC}\frac{dV}{dt} + \frac{1}{LC}V = 0$$

$$\frac{1}{RC} = \frac{1}{200\ 0{,}2\ 10^{-6}} = 25000$$

$$\frac{1}{LC} = \frac{1}{50\ 10^{-3}\ 0{,}2\ 10^{-6}} = 10^8$$

$$\frac{d^2V}{dt^2} + 25000\frac{dV}{dt} + 10^8 V = 0$$

$$\alpha = \frac{1}{2RC} = \frac{25000}{2} = 12500 \qquad \alpha^2 = 156\ 250\ 000 = 1{,}56\ 10^8$$

$$w_0^2 = \frac{1}{LC} = 10^8$$

roots are $$s_1 = -\alpha + \sqrt{\alpha^2 - w_0^2} = -12500 + \sqrt{12500^2 - 10^8}$$
$$= -5000$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - w_0^2} = -12500 - \sqrt{12500^2 - 10^8}$$
$$= -20\ 000$$

b) $\alpha^2 > w_0^2 \Rightarrow$ over damped

c) For $R = 312.5\ \Omega$,

$$\alpha = \frac{10^6}{(625)(0.2)} = 8000 \text{ rad/s},$$

$$\alpha^2 = 64 \times 10^6 = 0.64 \times 10^8 \text{ rad}^2/\text{s}^2.$$

As ω_0^2 remains at 10^8 rad^2/s^2,

$$s_1 = -8000 + j6000 \text{ rad/s},$$

$$s_2 = -8000 - j6000 \text{ rad/s}.$$

(In electrical engineering, the imaginary number $\sqrt{-1}$ is represented by the letter j, because the letter i represents current.)

In this case, the voltage response is underdamped since $\omega_0^2 > \alpha^2$.

d) For critical damping, $\alpha^2 = \omega_0^2$, so

$$\left(\frac{1}{2RC}\right)^2 = \frac{1}{LC} = 10^8,$$

or

$$\frac{1}{2RC} = 10^4,$$

and

$$R = \frac{10^6}{(2 \times 10^4)(0.2)} = 250\ \Omega.$$

Reminder

diff. equation	characteristic equation	solution of differential equ.
$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$	$s^2 + bs + c = 0$ $s_1 = -\frac{b}{2} + \frac{\sqrt{\Delta}}{2}$ $s_2 = -\frac{b}{2} - \frac{\sqrt{\Delta}}{2}$ $\Delta = b^2 - 4c$	$x(t) = d_1 e^{s_1 t} + d_2 e^{s_2 t}$
$\frac{dv^2}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$	$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$ $s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$ $s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$ $\alpha = \frac{1}{2RC}$ $\omega_0 = \frac{1}{\sqrt{LC}}$	$v(t) = d_1 e^{s_1 t} + d_2 e^{s_2 t}$

EXAMPLE 8.2

For the circuit in Fig. 8.6, $v(0^+) = 12$ V, and $i_L(0^+) = 30$ mA.

a) Find the initial current in each branch of the circuit.
b) Find the initial value of dv/dt.
c) Find the expression for $v(t)$.
d) Sketch $v(t)$ in the interval $0 \leq t \leq 250\ \mu$s.

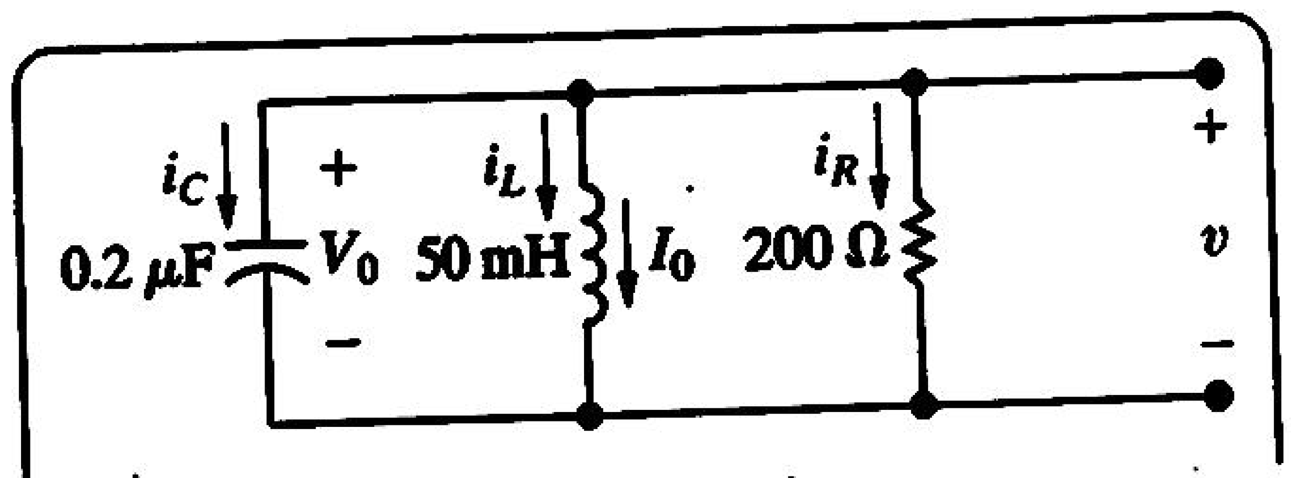


Figure 8.6 The circuit for Example 8.2.

SOLUTION

1) Inductor Current cannot change suddenly
2) capacitor Voltage Cannot change suddenly

$$I_L(0^+) = 30\,mA \qquad V(0^+) = 12^V$$

$$I_R = \frac{V}{R} \Rightarrow \quad I_R(0^+) = \frac{V(0^+)}{R} = \frac{12}{200\Omega} = 0.06A = 60\,mA$$

$$I_C + I_R + I_L = 0 \Rightarrow I_c(0^+) = -I_R(0^+) - I_L(0^+)$$

$$= -60 - 30 = -90\,mA$$

b) $$I_c = c\frac{dV_c}{dt} \Rightarrow \frac{dV_c}{dt} = \frac{I_c}{c}$$

$$\frac{dV_c(0^+)}{dt} = \frac{I_c(0^+)}{c} = \frac{-90 \times 10^{-3} A}{0.2\ 10^{-6} F} = -450000 \frac{V}{s}$$

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 200 \times 0.2\ 10^{-6}} = 12500$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{50\ 10^{-3}\ 0.2\ 10^{-6}}} = 10\,000$$

$$S_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -12500 + \sqrt{12500^2 - 10\,000^2} = -5000$$

$$S_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -12500 - \sqrt{12500^2 - 10000^2} = -20000$$

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$v(0) = A_1 e^0 + A_2 e^0 = 12^V \quad (1)$$

$$\frac{dv}{dt} = A_1 s_1 e^{s_1 t} + A_2 s_2 e^{s_2 t}$$

$$\frac{dv(0^+)}{dt} = A_1 s_1 e^0 + A_2 s_2 e^0 = 450\,000 \quad (2)$$

$$\left(\begin{array}{l} A_1 + A_2 = 12 \\ A_1(-5000) + A_2(-20000) = 450\,000 \end{array} \right) \Rightarrow \begin{array}{l} A_1 = -14 \\ A_2 = 26 \end{array}$$

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} = -14 e^{-5000 t} + 26 e^{-20000 t}$$

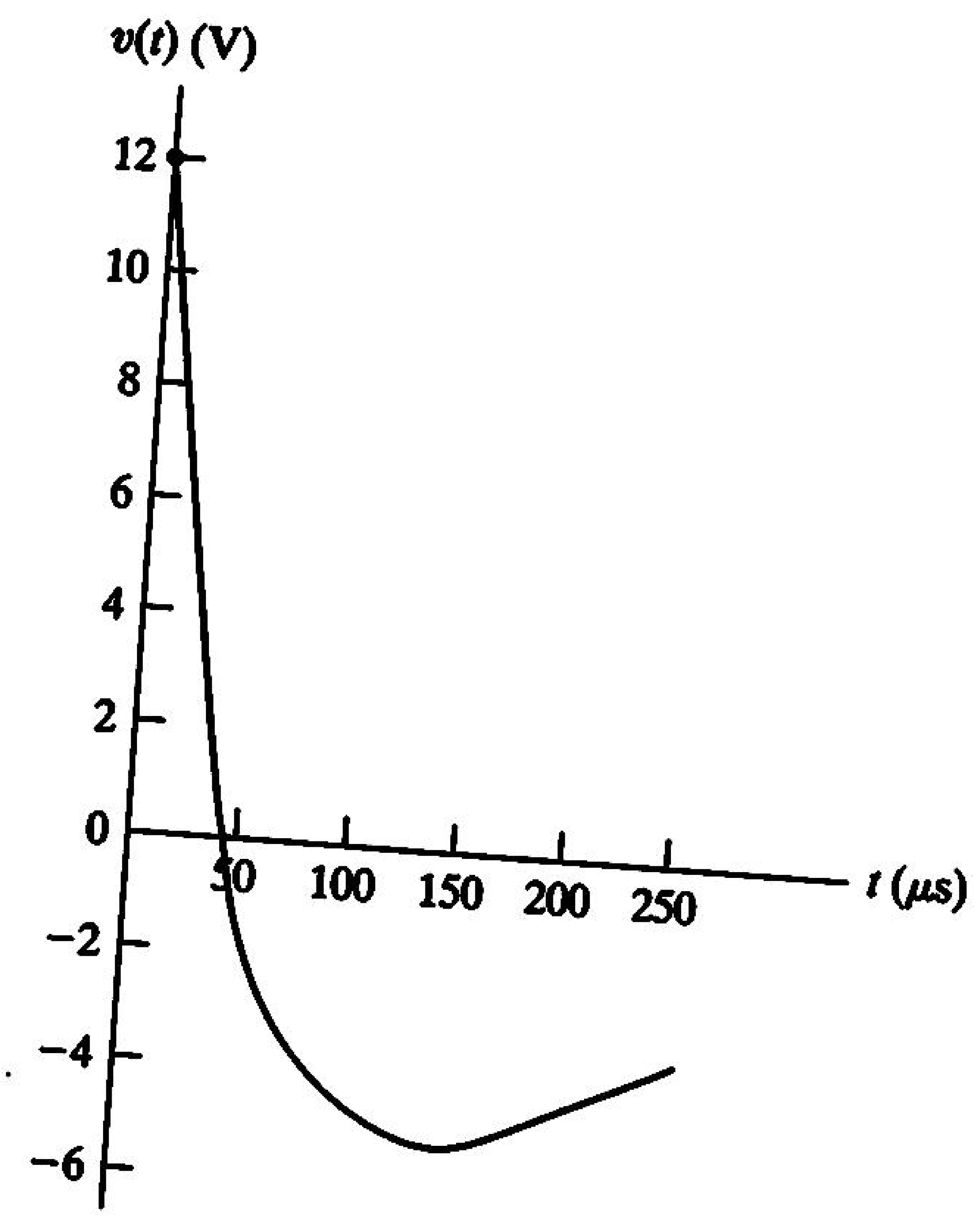
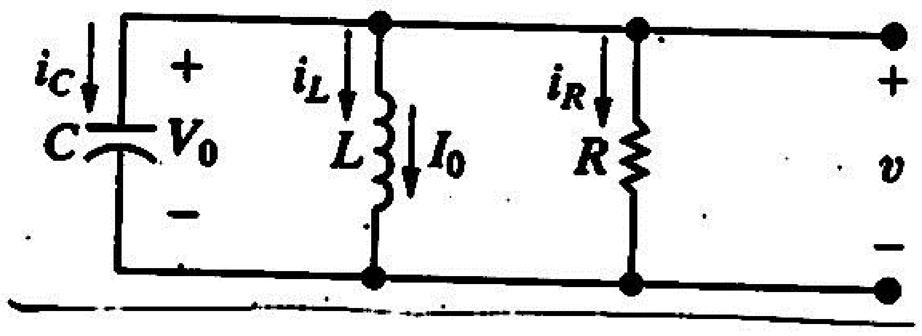


Figure 8.7 The voltage response for Example 8.2.



$V(t) = -14e^{-5000t} + 26e^{-20000t}$

8.3

Derive the expressions that describe the three branch currents i_R, i_L, and i_C in Example 8.2 (Fig. 8.6) during the time the stored energy is being released.

SOLUTION

We know the voltage across the three branches from the solution in Example 8.2, namely,

$$v(t) = (-14e^{-5000t} + 26e^{-20,000t})\text{ V}, \quad t \geq 0.$$

The current in the resistive branch is then

$$i_R(t) = \frac{v(t)}{200} = (-70e^{-5000t} + 130e^{-20,000t})\text{ mA}, \quad t \geq 0.$$

There are two ways to find the current in the inductive branch. One way is to use the integral relationship that exists between the current and the voltage at the terminals of an inductor:

$$i_L(t) = \frac{1}{L}\int_0^t v_L(x)dx + I_0.$$

A second approach is to find the current in the capacitive branch first and then use the fact that $i_R + i_L + i_C = 0$. Let's use this approach. Th current in the capacitive branch is

$$\begin{aligned} i_C(t) &= C\frac{dv}{dt} \\ &= 0.2 \times 10^{-6}(70,000e^{-5000t} - 520,000e^{-20,000t}) \\ &= (14e^{-5000t} - 104e^{-20,000t})\text{ mA}, \quad t \geq 0^+. \end{aligned}$$

Note that $i_C(0^+) = -90$ mA, which agrees with the result in Example 8.2.

Now we obtain the inductive branch current from the relationship

$$\begin{aligned} i_L(t) &= -i_R(t) - i_C(t) \\ &= (56e^{-5000t} - 26e^{-20,000t})\text{ mA}, \quad t \geq 0. \end{aligned}$$

We leave it to you, in Assessment Problem 8.2, to show that the integral relation alluded to leads to the same result. Note that the expression for i_L agrees with the initial inductor current, as it must.

Reminder

Solution of differential equation

roots are real $\Rightarrow$ $V(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

roots are complex $\Rightarrow$ $s_1 = a + bj$ $\quad s_2 = a - bj$

$$V(t) = (A_1 e^{s_1 t} + A_2 e^{s_2 t}) \equiv e^{at}(B_1 \cos bt + B_2 \sin bt)$$

$s_1 = -\alpha + \sqrt{\alpha^2 - w_0^2}$

if s_1 is complex $\quad s_1 = -\alpha + \sqrt{-(w_0^2 - \alpha^2)}$

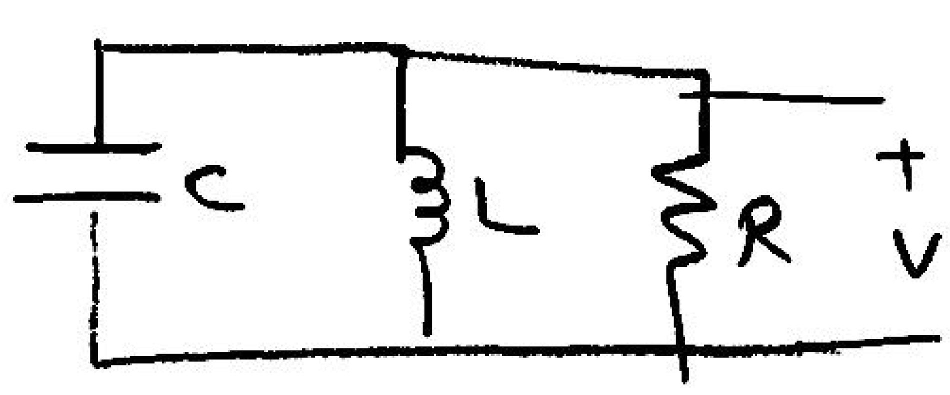
$$= -\alpha + \sqrt{-1}\sqrt{w_0^2 - \alpha^2}$$

$$= -\alpha + j\sqrt{w_0^2 - \alpha^2}$$

$$= -\alpha + j\,wd \qquad \boxed{wd = \sqrt{w_0^2 - \alpha^2}}$$

$s_2 = -\alpha - j\,wd$

Example problem write diff. equation, characteristic equation and find roots. $L = 0{,}2^H$ $\quad C = 1^F$ $\quad R = 0{,}5\,\Omega$



$\frac{1}{RC} = \frac{1}{0{,}5 \times 1} = 2 \qquad \frac{1}{LC} = \frac{1}{0{,}2 \times 1} = 5$

$$\frac{d^2V}{dt^2} + \frac{1}{RC}\frac{dV}{dt} + \frac{1}{LC}V \rightarrow \frac{d^2V}{dt^2} + 2\frac{dV}{dt} + 5V = 0$$

$s^2 + 2s + 5 = 0 \Rightarrow s_1 = -1 + 2j \qquad s_2 = -1 - 2j$

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or $\alpha = \frac{1}{2RC} = \frac{1}{2 \times 0{,}5 \times 1} = 1 \qquad w_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0{,}2 \times 1}} \qquad w_0^2 = 5$

$wd = \sqrt{w_0^2 - \alpha^2} = \sqrt{5 - 1} = 2$

$s_1 = -\alpha + j\,wd = -1 + 2j \qquad s_2 = -\alpha - j\,wd = -1 - 2j$

Example 8.4

In the circuit shown in Fig. 8.8, $V_0 = 0$, and $I_0 = -12.25$ mA.

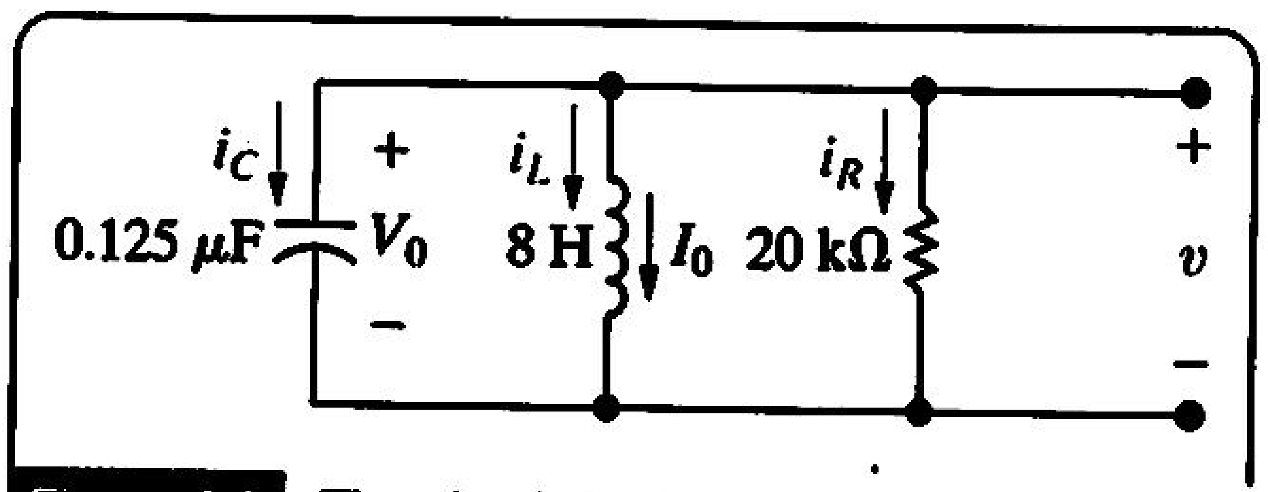


Figure 8.8 The circuit for Example 8.4.

a) Calculate the roots of the characteristic equation.
b) Calculate v and dv/dt at $t = 0^+$.
c) Calculate the voltage response for $t \geq 0$.
d) Plot $v(t)$ versus t for the time interval $0 \leq t \leq 11$ ms.

SOLUTION

a) Because

$$\alpha = \frac{1}{2RC} = \frac{10^6}{2(20)10^3(0.125)} = 200 \text{ rad/s},$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{\frac{10^6}{(8)(0.125)}} = 10^3 \text{ rad/s},$$

we have

$$\omega_0^2 > \alpha^2.$$

Therefore, the response is underdamped. Now,

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{10^6 - 4 \times 10^4} = 100\sqrt{96}$$
$$= 979.80 \text{ rad/s},$$

$$s_1 = -\alpha + j\omega_d = -200 + j979.80 \text{ rad/s},$$

$$s_2 = -\alpha - j\omega_d = -200 - j979.80 \text{ rad/s}.$$

b) $I_C + I_L + I_R = 0$

$V_c(0^+) = V_0 = 0$ (given)

$V_R(0^+) = V_c(0^+) = 0$

$$I_R(0^+) = \frac{V_R(0^+)}{R} = 0$$

$I_L(0^+) = I_0 = -12.25$ mA (given)

$I_c(0^+) + I_L(0^+) + I_R(0^+) = 0$

$I_c(0^+) - 12.25 + 0 = 0$

$I_c(0^+) = 12.25$ mA

$$I_c = C\frac{dV_c}{dt}$$

$$I_c(0^+) = C\left.\frac{dV_c}{dt}\right|_{t=0^+} = 12.25$$

$$\left.\frac{dV_c}{dt}\right|_{t=0^+} = \frac{12.25}{C} = \frac{12.25 \times 10^{-3} A}{0.125 \times 10^{-6} F}$$

$$= 98\,000 \ \frac{V}{s}$$

c) $v(t) = e^{-200t}(B_1 \cos 979t + B_2 \sin 979t)$

$V(0) = 0 \Rightarrow e^0(B_1 \cos 0 + B_2 \sin 0) = 0$

$B_1 + 0 = 0 \Rightarrow B_1 = 0$

$v(t) = e^{-200t} B_2 \sin 979t$

$$\frac{dv}{dt} = -200\, e^{-200t} B_2 \sin 979t + e^{-200t} B_2\, 979 \cos 979t$$

$$\left.\frac{dv}{dt}\right|_{t=0} = 98000 = 0 + e^0 B_2\, 979$$

$$B_2 = \frac{98000}{979} \cong 100$$

$v(t) = 100\, e^{-200t} \sin 979$

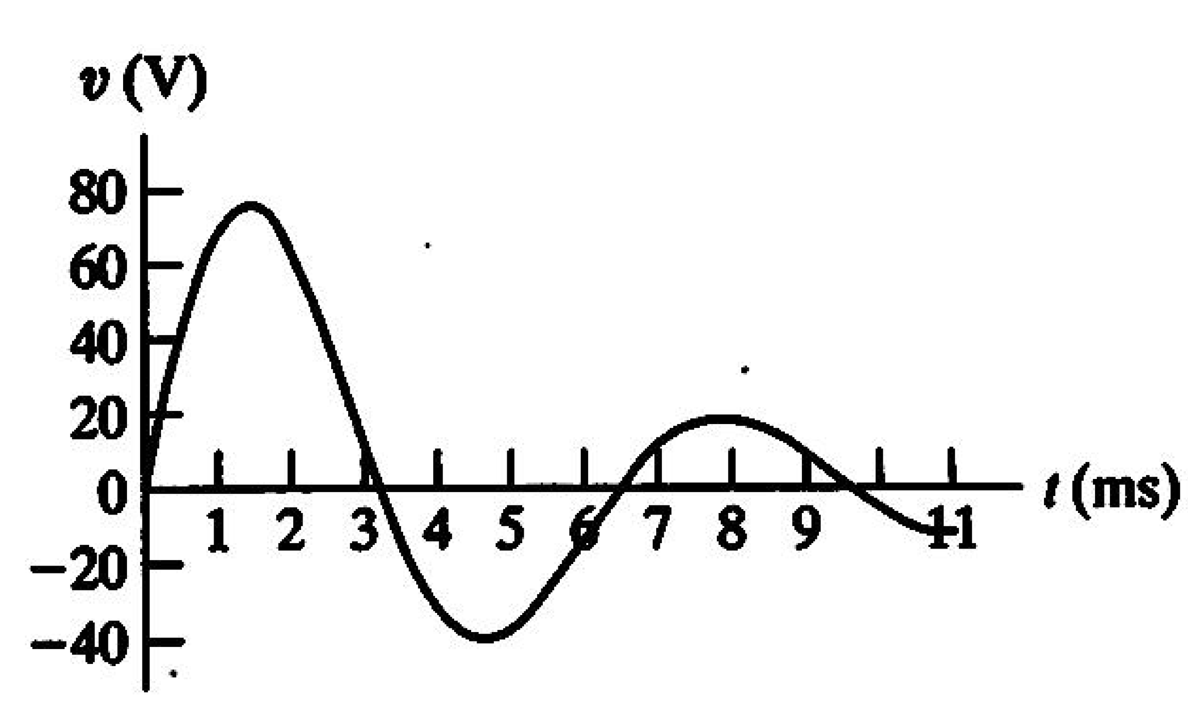
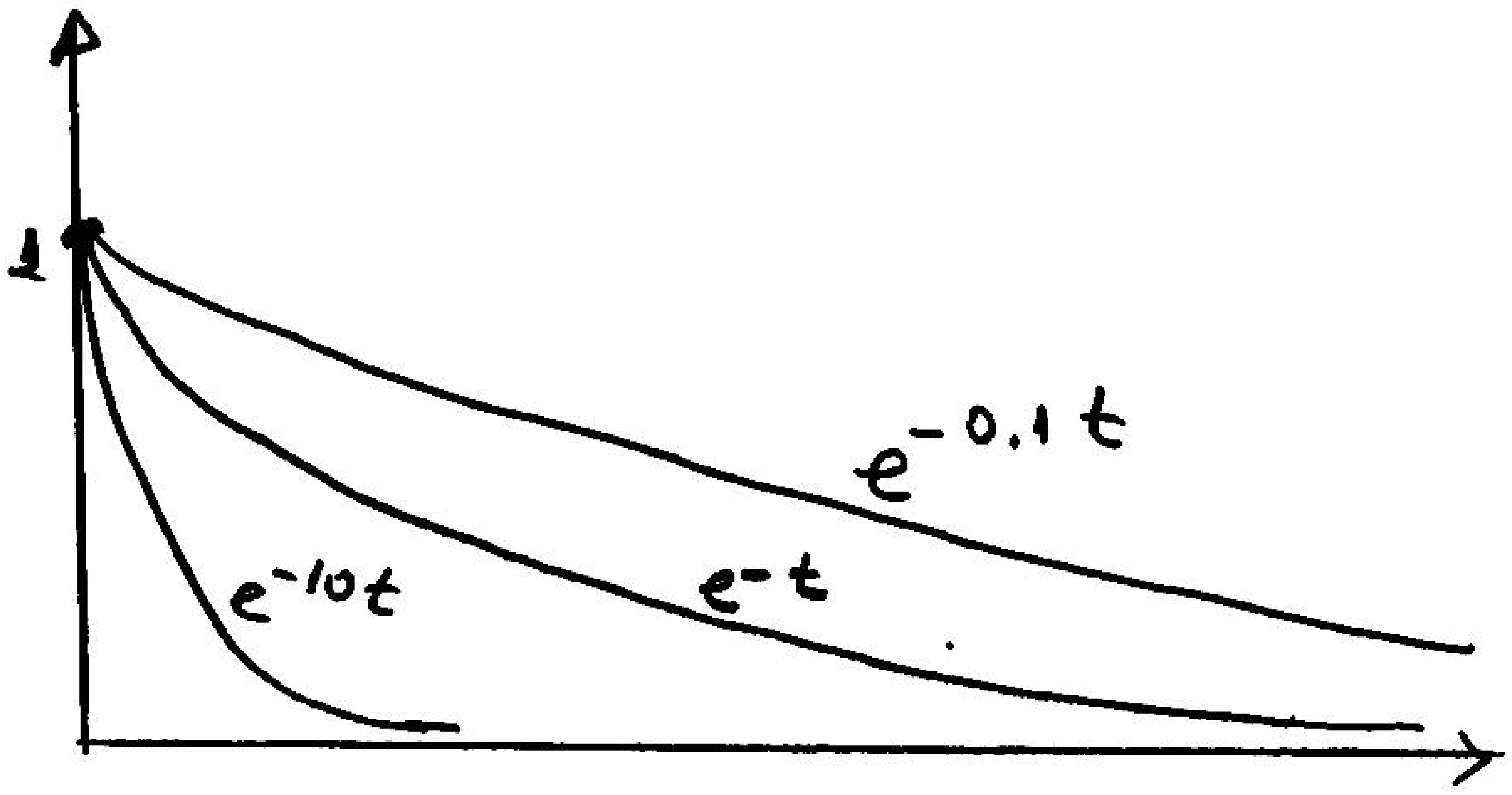
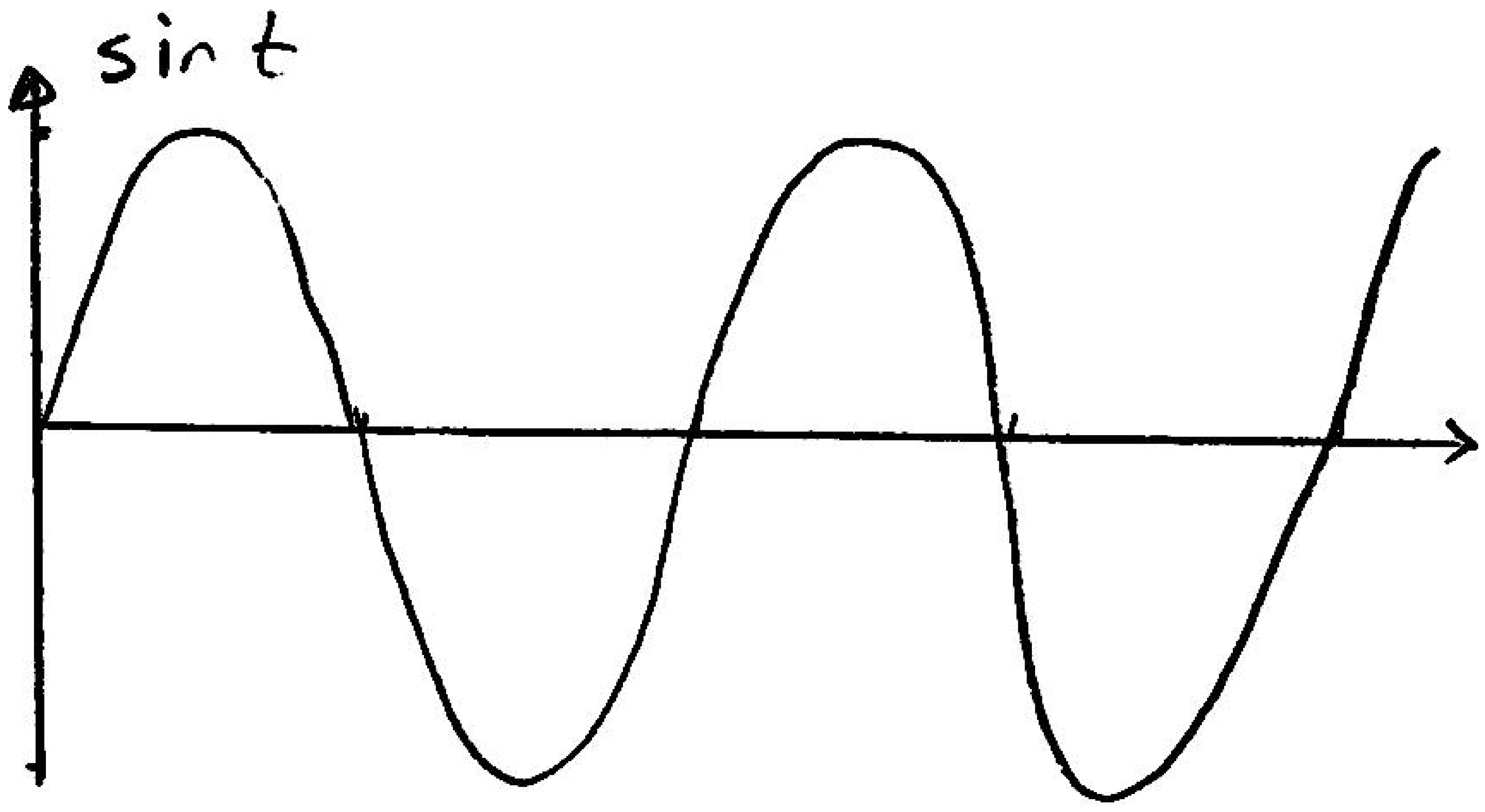
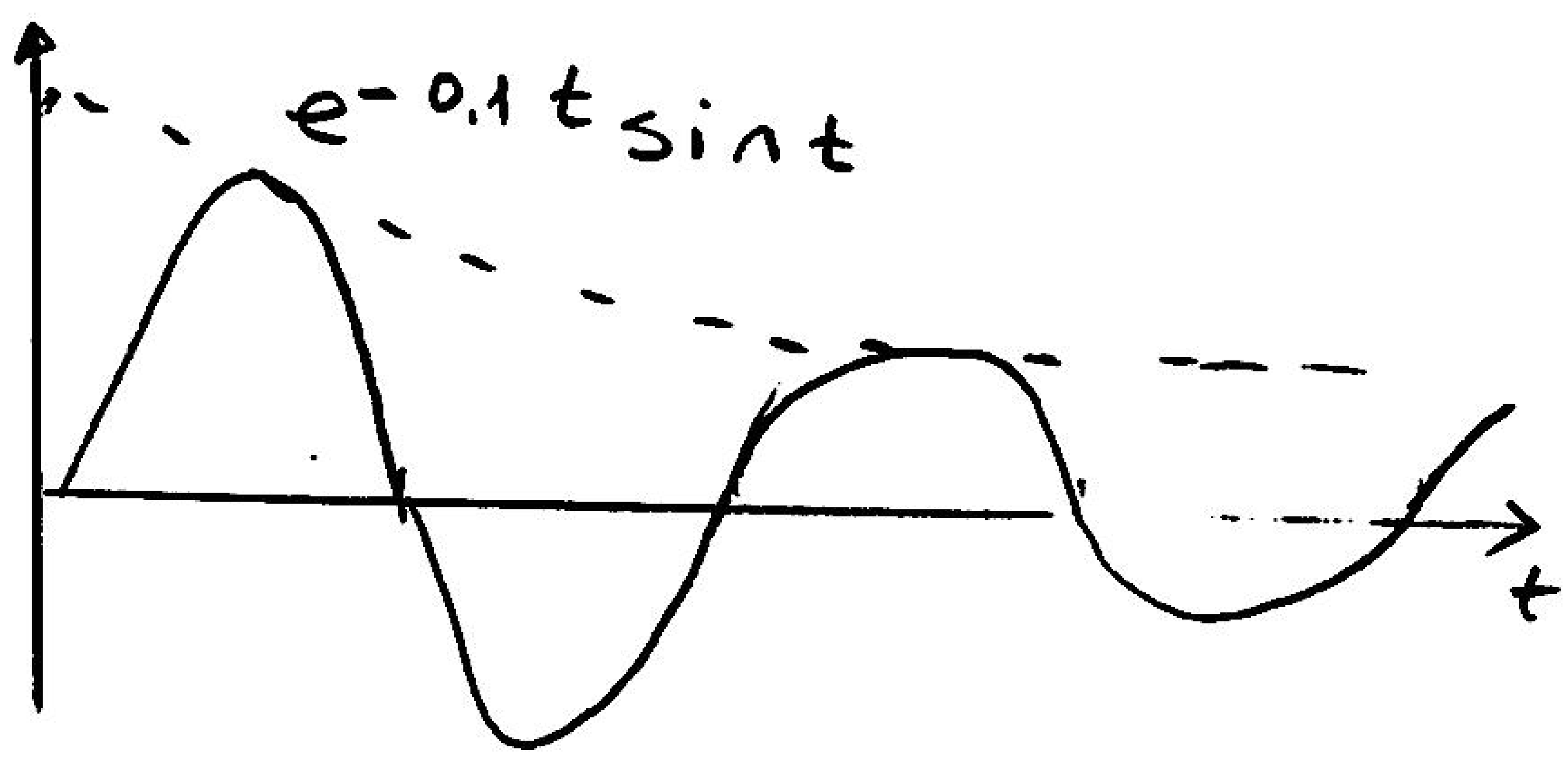


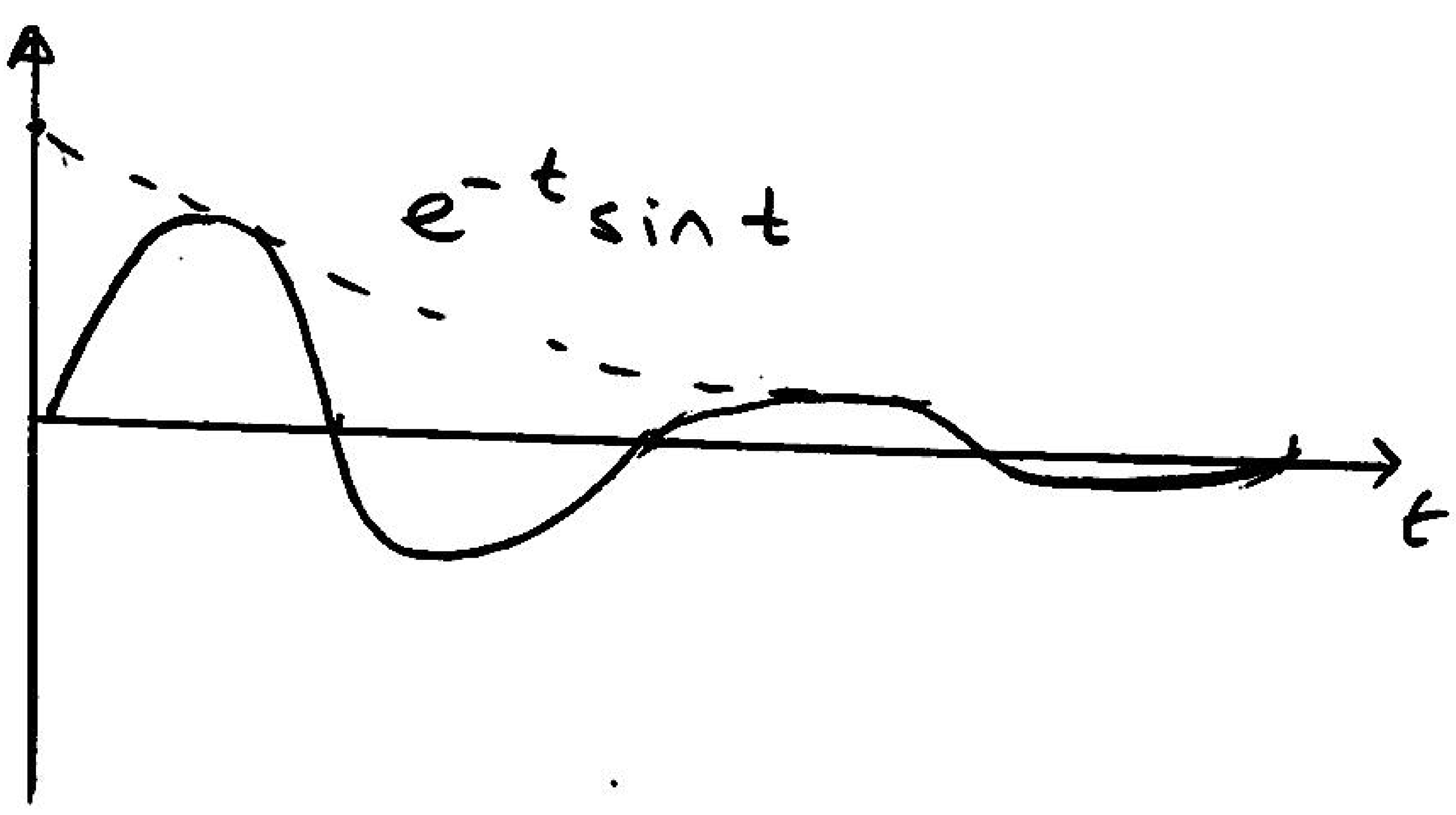
Figure 8.9 The voltage response for Example 8.4.

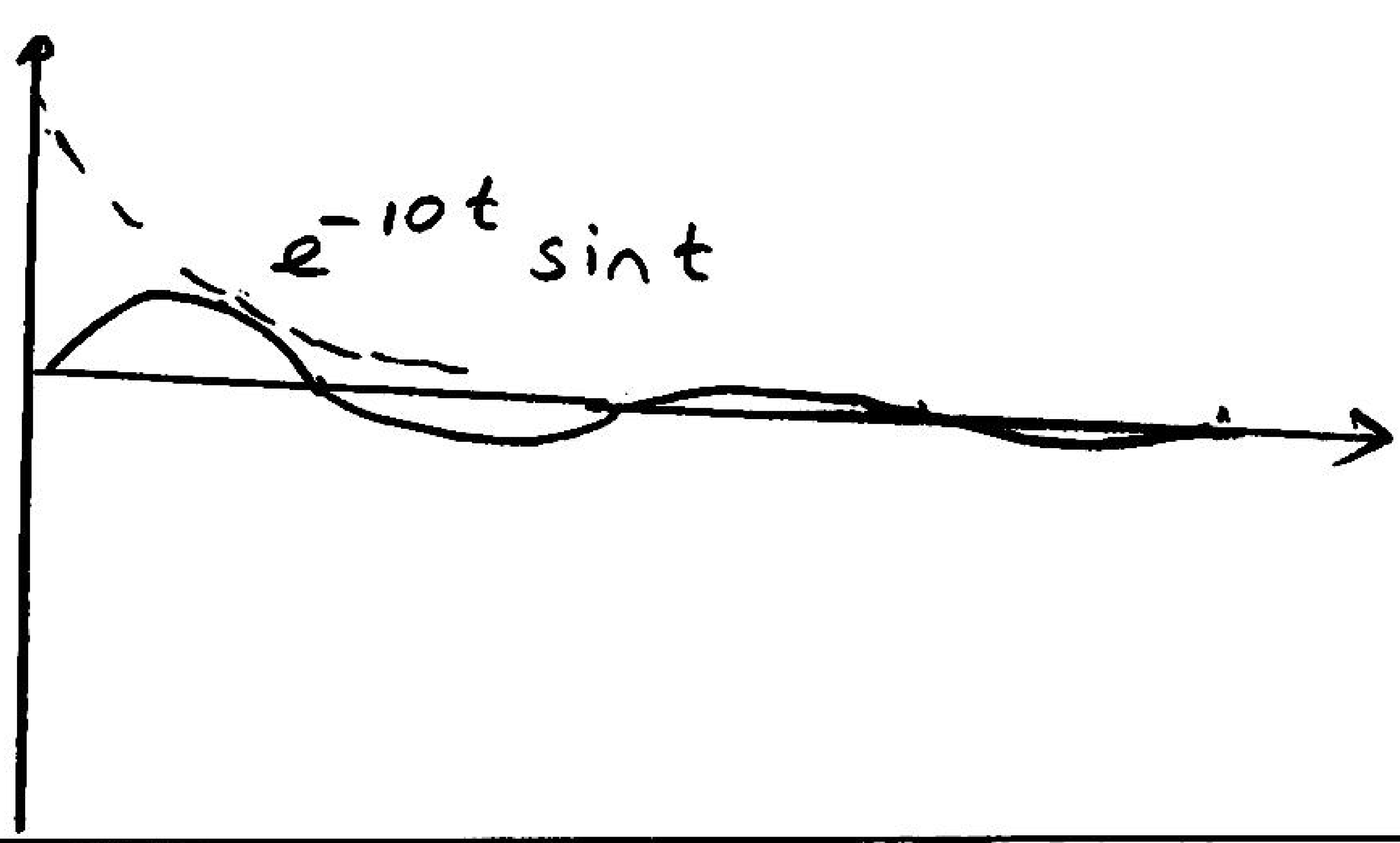
Graphics

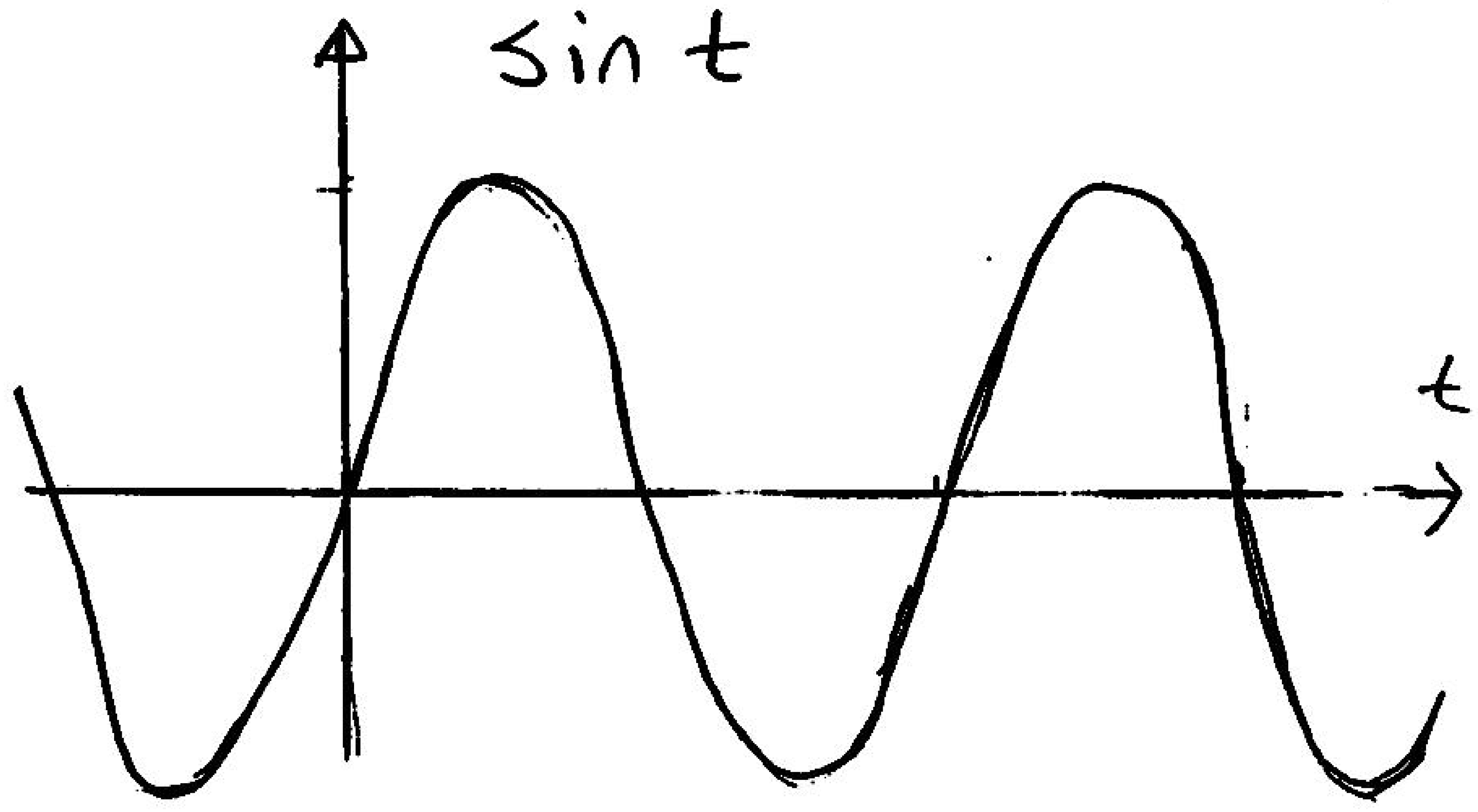


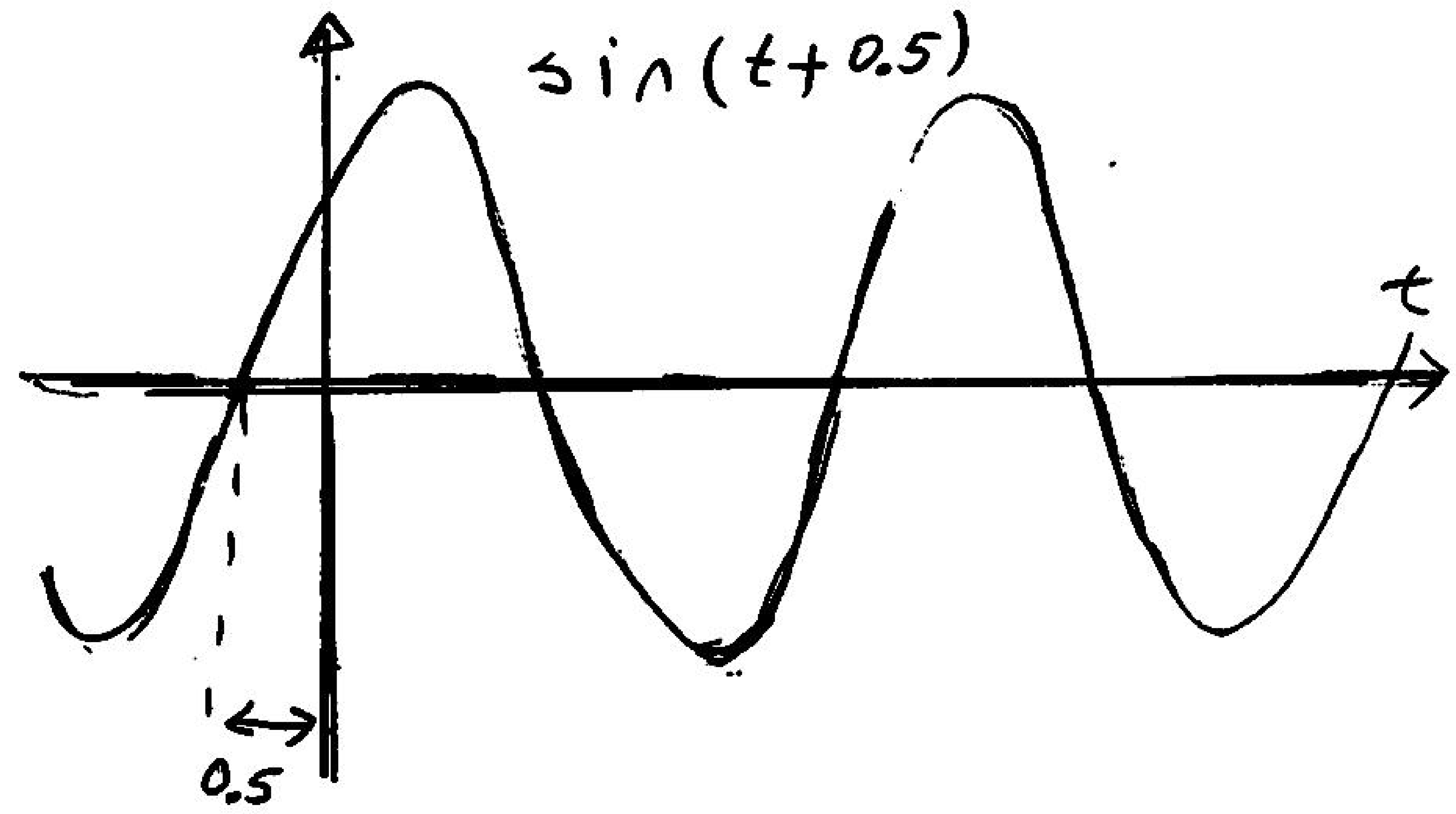


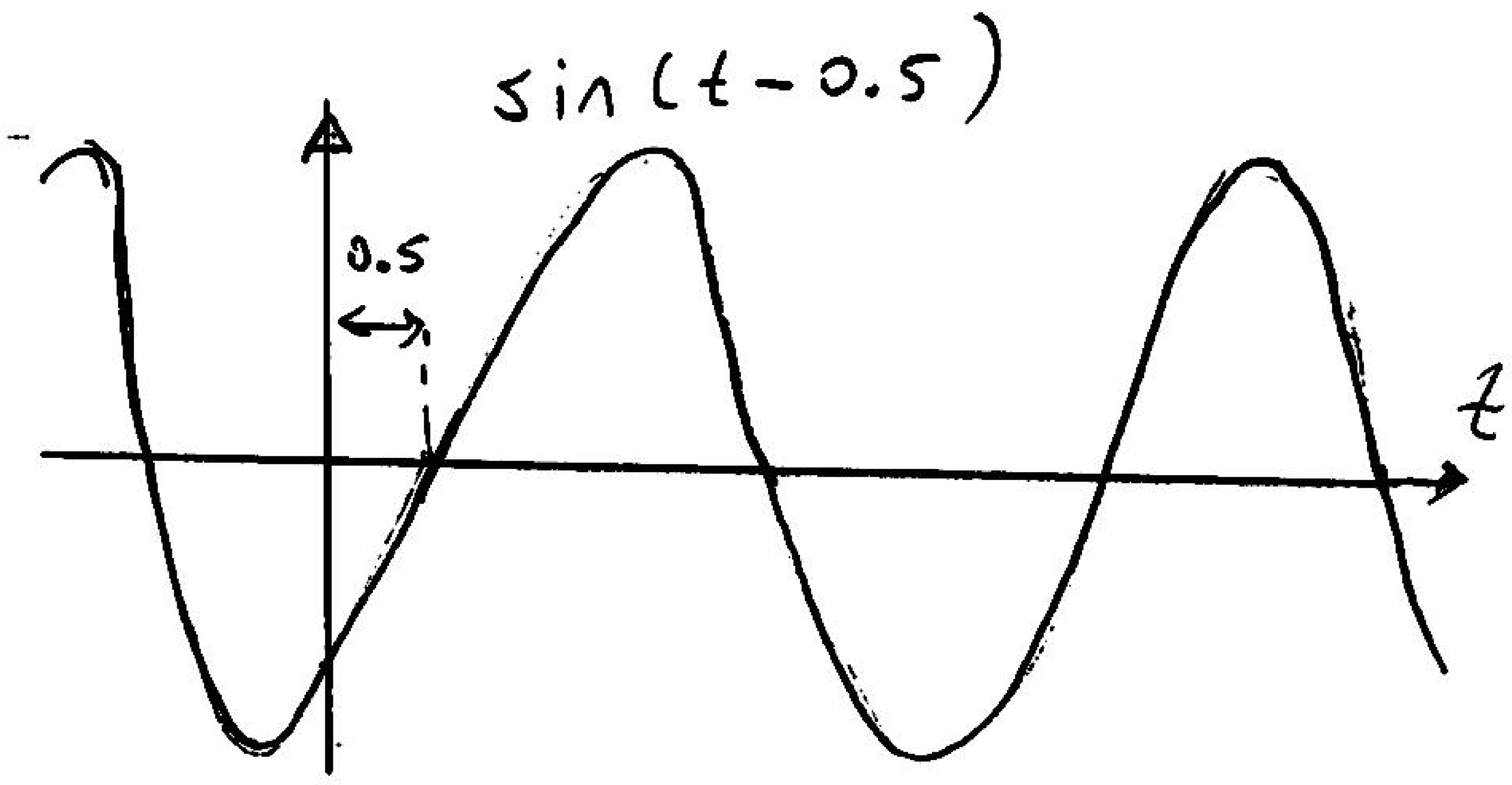










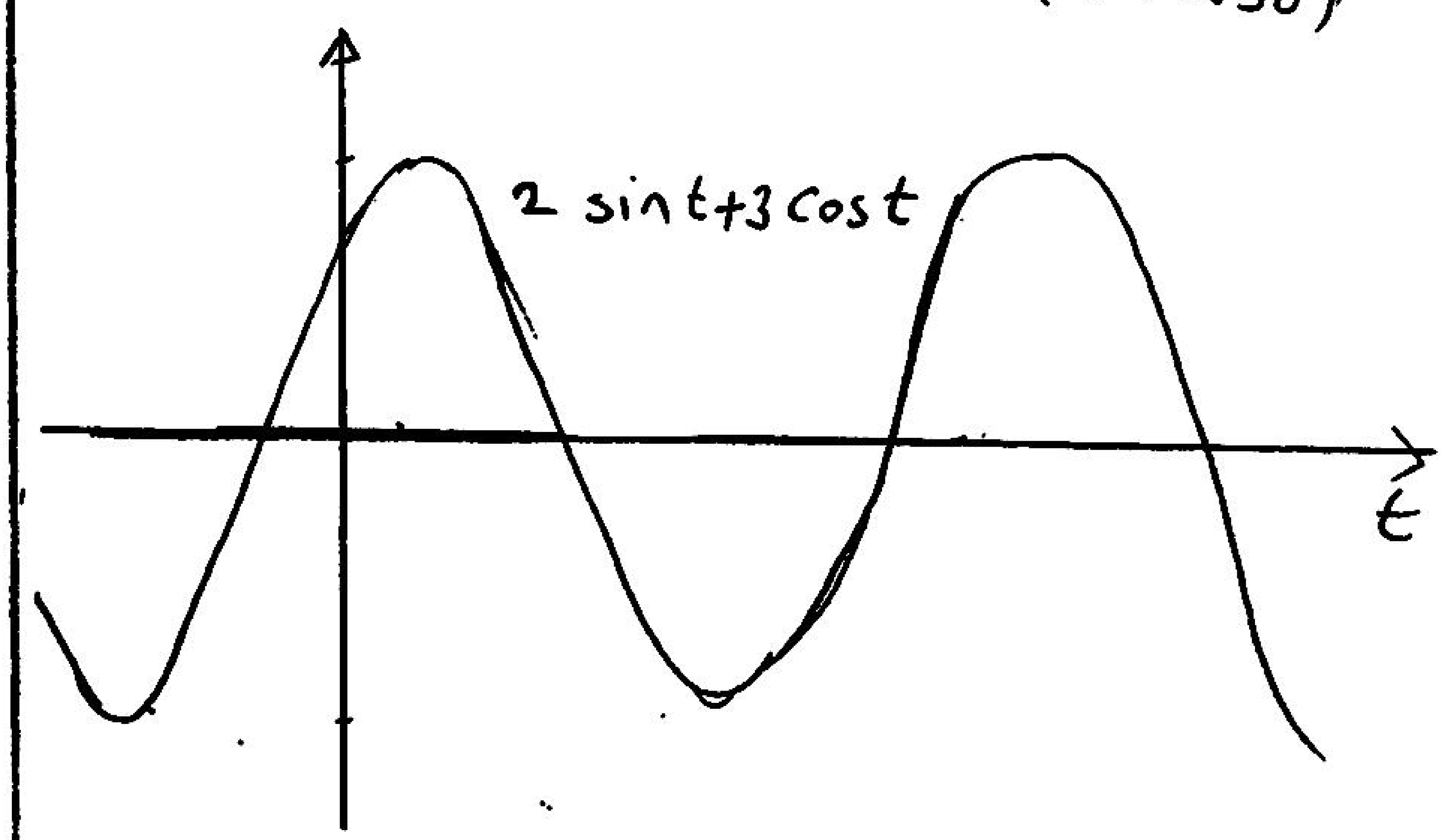


$$A\cos t + B\sin t = \sqrt{A^2+B^2}\,\sin(t+\theta)$$

$$\theta = \tan^{-1}\frac{B}{A}$$

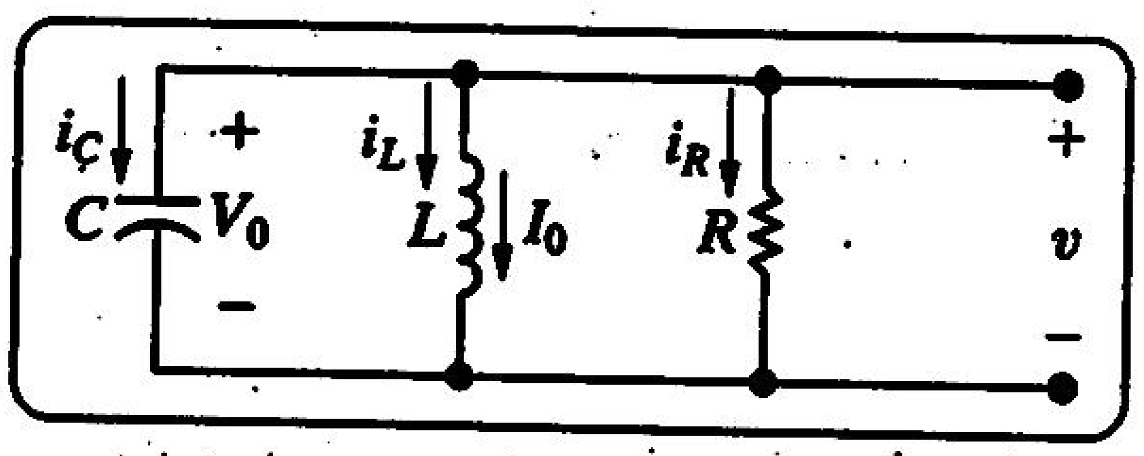
radian

$$2\cos t + 3\sin t = 3.6\,\sin(t+0.98)$$



◆ **Be able to determine the natural and the step response of parallel *RLC* circuits**

8.4 A 10 mH inductor, a 1 μF capacitor, and a variable resistor are connected in parallel in the circuit shown. The resistor is adjusted so that the roots of the characteristic equation are $-8000 \pm j6000$ rad/s. The initial voltage on the capacitor is 10 V, and the initial current in the inductor is 80 mA. Find (a) R; (b) $dv(0^+)/dt$; (c) B_1 and B_2 in the solution for v; and (d) $i_L(t)$.



ANSWER: (a) 62.5 Ω; (b) −240,000 V/s; (c) $B_1 = 10$ V, $B_2 = -80/3$ V; (d) $i_L(t) = 10e^{-8000t}[8\cos 6000t + (82/3)\sin 6000t]$ mA when $t \geq 0$.

solution:

a) $S_1 = -\alpha + j w_d$ $\qquad \alpha = \dfrac{1}{2RC}$ $\qquad w_0 = \dfrac{1}{\sqrt{LC}}$ $\qquad w_d = \sqrt{w_0^2 - \alpha^2}$

$S_1 = -8000 + j6000$ $\qquad 8000 = \dfrac{1}{2 \times 1 \times 10^{-6} R} \Rightarrow R = \dfrac{1}{8000 \times 2 \times 10^{-6}} = 62.5$

b) $V_c(0^+) = 10^V = V_R(0^+) \Rightarrow I_R(0^+) = \dfrac{10}{R} = \dfrac{10}{62.5} = 0.16\ A$

$I_c + I_L + I_R = 0 \Rightarrow I_c(0^+) = -I_L(0^+) - I_R(0^+) = -80_{mA} - 0.16\ A$

$= -0.08 - 0.16 = -0.24\ A$

$I_c = c\dfrac{dV_c}{dt} \Rightarrow \left.\dfrac{dV_c}{dt}\right|_{t=0^+} = \dfrac{I_c(0^+)}{c} = \dfrac{-0.24}{1 \times 10^{-6}} = -0.24\ 10^6 \dfrac{V}{S}$

$= -240\ 000 \dfrac{V}{S}$

$V(t) = e^{-8000t}(B_1 \cos 6000t + B_2 \sin 6000t)$

$V(0) = 10 = e^0(B_1 \cos 0 + B_2 \sin 0) \Longrightarrow B_1 = 10$

$\dfrac{dV}{dt} = -8000e^{-8000t}(B_1 \cos 6000t + B_2 \sin 6000t)$

$+ e^{-8000t}(-B_1\, 6000 \sin 6000t + B_2\, 6000 \cos 6000t)$

$\left.\dfrac{dV}{dt}\right|_{t=0} = -240\ 000 = -8000\, B_1 + 6000\, B_2 \Rightarrow B_2 = -\dfrac{80}{3} = -26.6$

$I_L(t) = -I_c(t) - I_R(t) = -c\dfrac{dV}{dt} - \dfrac{V}{R} = \ldots\ldots$

The Critically Damped Voltage Response

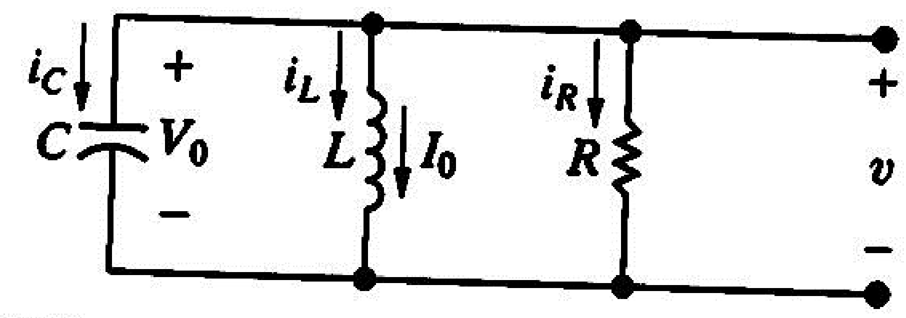


Figure 8.1 A circuit used to illustrate the natural response of a parallel *RLC* circuit.

$$\Rightarrow \frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0.$$

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \rightarrow \begin{cases} \text{real roots} \\ \text{complex roots} \\ \text{equal real roots} \end{cases}$$

real roots $\Rightarrow$ overdamped $\Rightarrow v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

complex roots $(s = a \mp bi) \Rightarrow$ Underdamped $\Rightarrow v(t) = e^{at}(B_1 \cos bt + B_2 \sin bt)$

equal roots $\Rightarrow$ critically damped $\Rightarrow v(t) = D_1 t e^{st} + D_2 e^{st}$

equal roots $s = -\alpha + \underbrace{\sqrt{\alpha^2 - \omega_0^2}}_{0}$ $\quad \boxed{\alpha = \omega_0}$

$$\boxed{v(0) = D_1 \times 0\, e^0 + D_2 e^0 = D_2} \qquad \boxed{s = -\alpha}$$

$$\frac{dv}{dt} = D_1 e^{st} + D_1 s t e^{st} + D_2 s e^{st}$$

$$\left.\frac{dv}{dt}\right|_{t=0} = D_1 + D_2 s = D_1 - \alpha D_2 \quad (8.36)$$

8.5

a) For the circuit in Example 8.4 (Fig. 8.8), find the value of R that results in a critically damped voltage response.

b) Calculate $v(t)$ for $t \geq 0$.

c) Plot $v(t)$ versus t for $0 \leq t \leq 7$ ms.

SOLUTION

a) From Example 8.4, we know that $\omega_0^2 = 10^6$. Therefore for critical damping, $\alpha = \omega_0$

$$\alpha = 10^3 = \frac{1}{2RC},$$

or

$$R = \frac{10^6}{(2000)(0.125)} = 4000\ \Omega.$$

b) From the solution of Example 8.4, we know that $v(0^+) = 0$ and $dv(0^+)/dt = 98{,}000$ V/s. From Eqs. 8.35 and 8.36, $D_2 = 0$ and $D_1 = 98{,}000$ V/s. Substituting these values for α, D_1, and D_2 into Eq. 8.34 gives

$$v(t) = 98{,}000te^{-1000t}\ \text{V}, \quad t \geq 0.$$

c) Figure 8.10 shows a plot of $v(t)$ versus t in the interval $0 \leq t \leq 7$ ms.

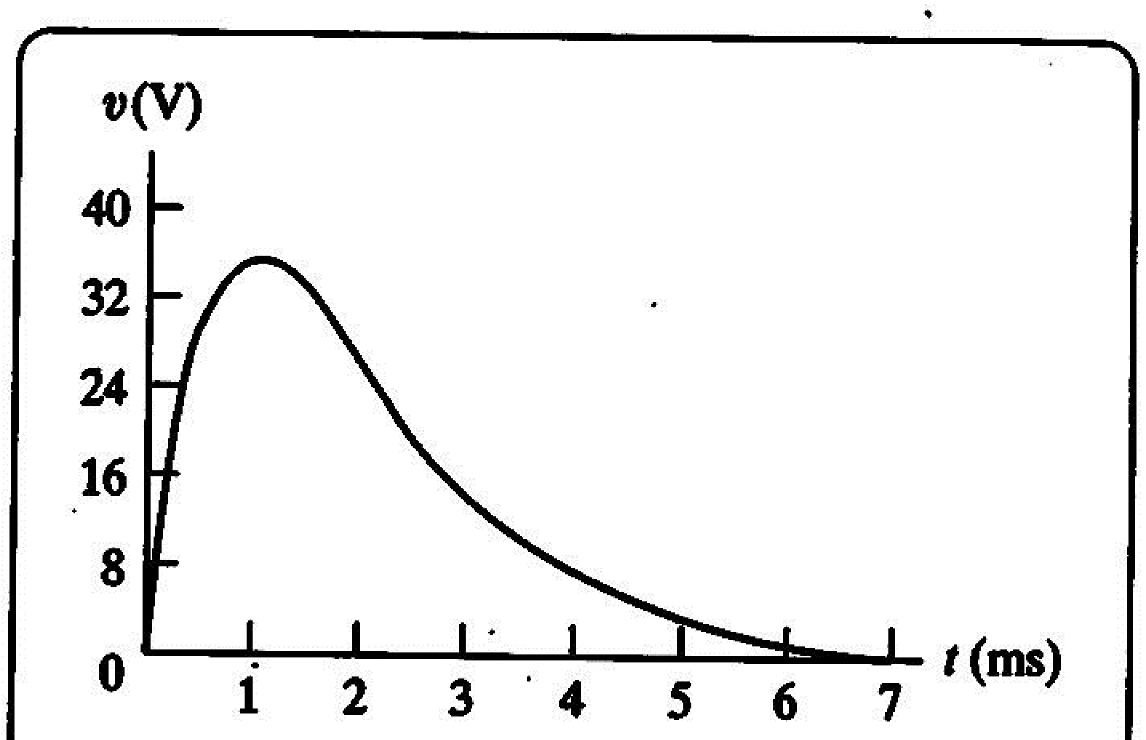


Figure 8.10 The voltage response for Example 8.5.