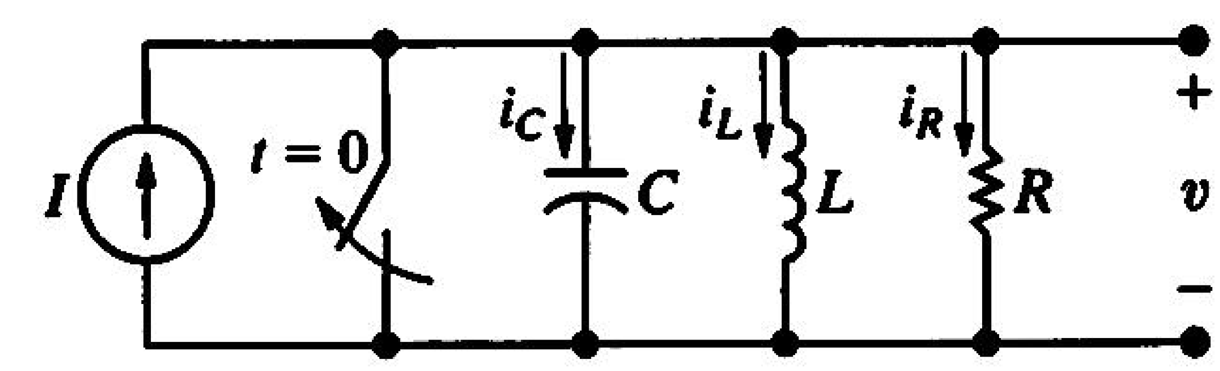


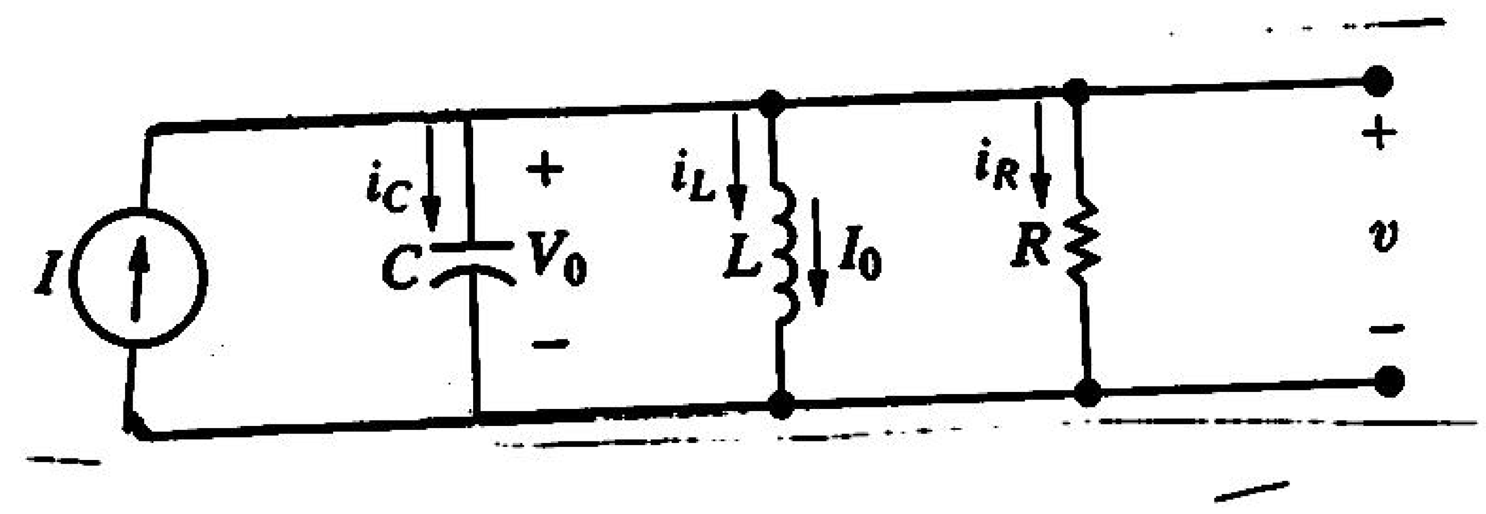


Figure 8.11 A circuit used to describe the step response of a parallel *RLC* circuit.

$$I = i_c + i_L + i_R$$

$$i_c = C\frac{dV_c}{dt} \qquad i_R = \frac{V_R}{R} \qquad V_L = L\frac{di_L}{dt} \Rightarrow \frac{dV_L}{dt} = L\frac{d^2 i_L}{dt^2}$$

$$V_c = V_R = V_L = V$$

$$I = C\frac{dV}{dt} + i_L + \frac{1}{R}V$$

$$I = C\left(L\frac{d^2 i_L}{dt^2}\right) + i_L + \frac{1}{R}\left(L\frac{di_L}{dt}\right)$$

$$LC\frac{d^2 i_L}{dt^2} + \frac{L}{R}\frac{di_L}{dt} + i_L = I$$

$$\boxed{\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{1}{LC}i_L = \frac{1}{LC}I}$$

Total solution = Natural response + forced sitep response

Natural response $I = 0 \Rightarrow i_L(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

forced step response $\Rightarrow \frac{di_L}{dt} = 0 \Rightarrow \boxed{i_L = I}$

$$\boxed{i_L(t) = I + A_1 e^{s_1 t} + A_2 e^{s_2 t}}$$

EXAMPLE 8.6

The initial energy stored in the circuit in Fig. 8.12 is zero. At $t = 0$, a dc current source of 24 mA is applied to the circuit. The value of the resistor is 400 Ω.

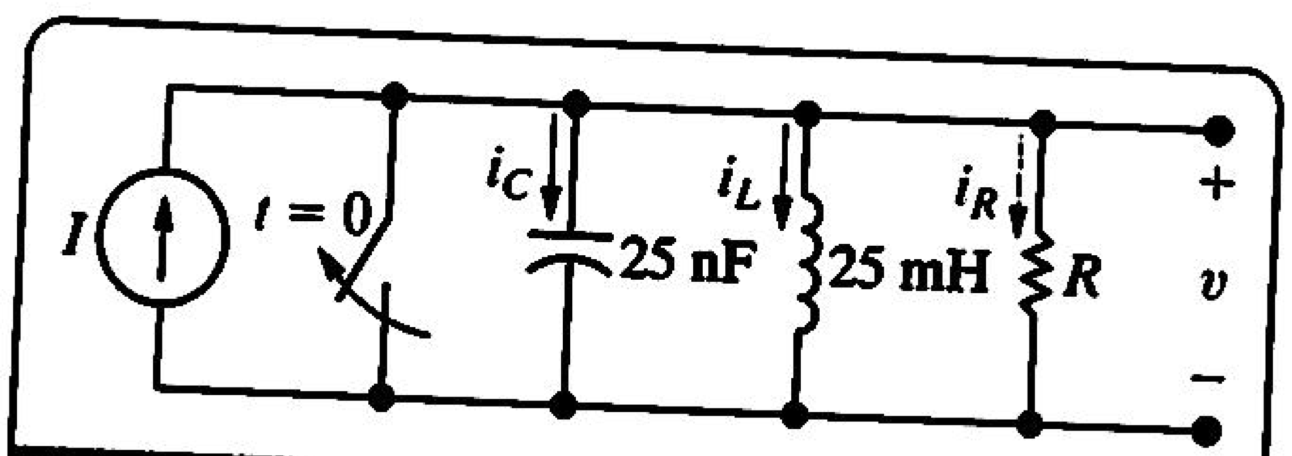


Figure 8.12 The circuit for Example 8.6.

a) What is the initial value of i_L?

b) What is the initial value of di_L/dt?

c) What are the roots of the characteristic equation?

d) What is the numerical expression for $i_L(t)$ when $t \geq 0$?

SOLUTION

a) No energy is stored in the circuit means

$i_L(0^+) = 0 \qquad V_c(0^+) = 0$

b) $V_L = L \dfrac{di_L}{dt} \Rightarrow \left.\dfrac{di_L}{dt}\right|_{t=0} = \dfrac{V_L(0^+)}{L} = 0$

$V_L = V_c$ (always)

c) From the circuit elements, we obtain

$$\omega_0^2 = \frac{1}{LC} = \frac{10^{12}}{(25)(25)} = 16 \times 10^8,$$

$$\alpha = \frac{1}{2RC} = \frac{10^9}{(2)(400)(25)} = 5 \times 10^4 \text{ rad/s},$$

or

$$\alpha^2 = 25 \times 10^8.$$

Because $\omega_0^2 < \alpha^2$, the roots of the characteristic equation are real and distinct. Thus

$$s_1 = -5 \times 10^4 + 3 \times 10^4 = -20{,}000 \text{ rad/s},$$

$$s_2 = -5 \times 10^4 - 3 \times 10^4 = -80{,}000 \text{ rad/s}.$$

d) Because the roots of the characteristic equation are real and distinct, the inductor current response will be overdamped. Thus $i_L(t)$ takes the form of Eq. 8.47, namely,

$$i_L = \underset{\downarrow}{I_f} + A_1' e^{s_1 t} + A_2' e^{s_2 t}.$$

$I = 24$ mA

$i_L(0^+) = 0 = 24 + A_1 e^0 + A_2 e^0$

$\boxed{A_1 + A_2 = -24}$

$\dfrac{di_L}{dt} = 0 + A_1 s_1 e^{s_1 t} + A_2 s_2 e^{s_2 t}$

$\left.\dfrac{di_L}{dt}\right|_{t=0} = 0 = A_1 s_1 + A_2 s_2$

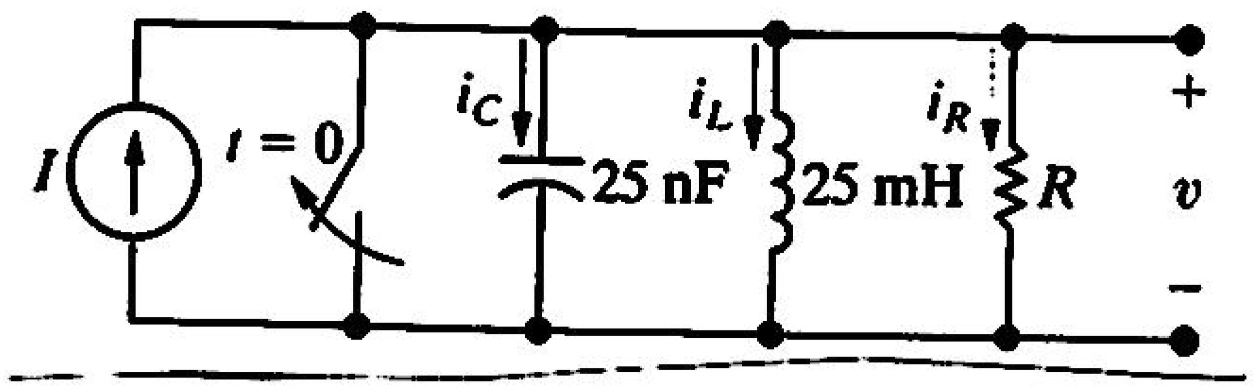
$\boxed{-20000 A_1 - 80000 A_2 = 0}$

$\left(\begin{array}{l} A_1 + A_2 = -24 \\ -20000 A_1 - 80000 A_2 = 0 \end{array}\right) \Rightarrow \begin{array}{l} A_1 = -32 \\ A_2 = 8 \end{array}$

$i_L = \left(24 - 32 e^{-20000t} + 8 e^{-80000t}\right)$ mA

EXAMPLE 8.7

The resistor in the circuit in Example 8.6 (Fig. 8.12) is increased to 625 Ω. Find $i_L(t)$ for $t \geq 0$.



$R = 625 \quad I_L(0) = 0 \quad V_c(0) = 0$

SOLUTION

$$\omega_0^2 = \frac{1}{LC} = \frac{10^{12}}{(25)(25)} = 16 \times 10^8,$$

$$\alpha = \frac{1}{2RC} = \frac{1}{625 \; 25 \; 10^{-9}} = 32000$$

$$w_d = \sqrt{w_0^2 - \alpha^2} = \sqrt{16 \times 10^8 - 32000^2}$$

$$= 24000$$

$$s_1 = -\alpha + j w_d = -32000 + j\,24000$$

$$s_2 = -\alpha - j w_d = -32000 - j\,24000$$

The current response is now underdamped and given by Eq. 8.48:

$$i_L(t) = I_f + B_1' e^{-\alpha t} \cos \omega_d t + B_2' e^{-\alpha t} \sin \omega_d t.$$

$$\frac{di_L}{dt} = 0 + B_1'\left[-\alpha e^{-\alpha t} \cos w_d t + e^{-\alpha t} w_d(-\sin w_d t)\right]$$
$$+ B_2'\left[-\alpha e^{-\alpha t} \sin w_d t + e^{-\alpha t} w_d \cos w_d t\right]$$

$$i_L(0) = I_f + B_1' = 0,$$

$$\frac{di_L}{dt}(0) = \omega_d B_2' - \alpha B_1' = 0.$$

Then,

$$B_1' = -24 \text{ mA}$$

and

$$B_2' = -32 \text{ mA}.$$

The numerical solution for $i_L(t)$ is

$$i_L(t) = (24 - 24e^{-32{,}000t} \cos 24{,}000t$$
$$- 32e^{-32{,}000t} \sin 24{,}000t) \text{ mA}, \qquad t \geq 0.$$

EXAMPLE 8.8

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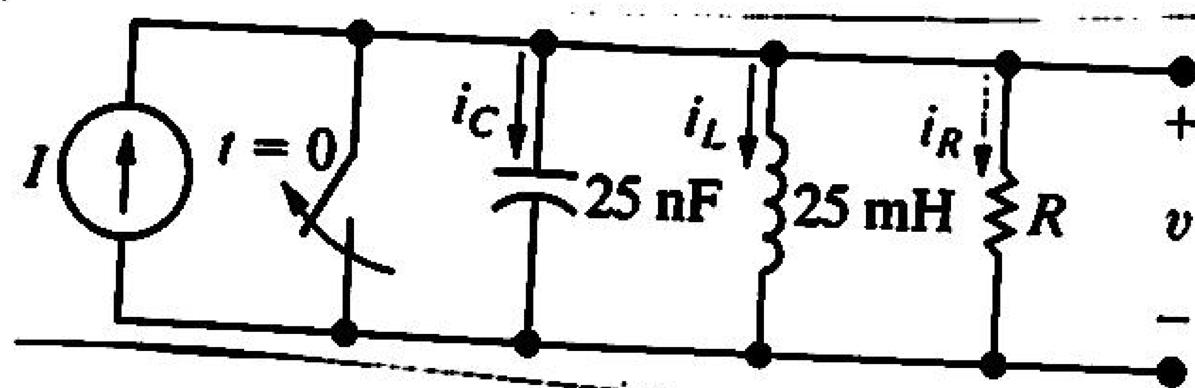
The resistor in the circuit in Example 8.6 (Fig. 8.12) is set at 500 Ω. Find i_L for $t \geq 0$.



$R = 500\Omega \quad I_L(0) = 0 \quad V_c(0) = 0$

SOLUTION

$$\omega_0^2 = \frac{1}{LC} = \frac{10^{12}}{(25)(25)} = 16 \times 10^8$$

$$\alpha = \frac{1}{2RC} = \frac{1}{500 \times 25 \times 10^{-9}} = 40\,000$$

$$\alpha^2 = 16 \; 10^8$$

$$w^2 = \alpha^2 \Rightarrow \text{critical damp.}$$

$$i_L(t) = I_f + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t}.$$

$$\frac{di_L}{dt} = 0 + D_1'\left[e^{-\alpha t} - \alpha t e^{-\alpha t}\right] - \alpha D_2' e^{-\alpha t}$$

Again, D_1' and D_2' are computed from initial conditions, or

$$i_L(0) = \overset{24}{I_f} + D_2' = 0, \quad \Rightarrow D_2' = -24$$

$$\frac{di_L}{dt}(0) = D_1' - \alpha D_2' = 0.$$

$$D_1' = \alpha D_2'$$
$$= 40000\,(-24\text{mA})$$

Thus

$$D_1' = -960{,}000 \text{ mA/s} \quad \text{and} \quad D_2' = -24 \text{ mA}.$$

The numerical expression for $i_L(t)$ is

$$i_L(t) = (24 - 960{,}000te^{-40{,}000t} - 24e^{-40{,}000t}) \text{ mA}, \quad t \geq 0.$$

EXAMPLE 8.10

Energy is stored in the circuit in Example 8.8 (Fig. 8.12, with $R = 500\ \Omega$) at the instant the dc current source is applied. The initial current in the inductor is 29 mA, and the initial voltage across the capacitor is 50 V. Find (a) $i_L(0)$; (b) $di_L(0)/dt$; (c) $i_L(t)$ for $t \geq 0$; (d) $v(t)$ for $t \geq 0$.

Solution

Everything is the same

except $i_L(0^+) = 29\ mA$

$V_c(0^+) = 50^V$

$V_L = L\frac{di_L}{dt}$ $\quad \frac{di_L}{dt} = \frac{V_L}{L}$

$$\frac{di_L(0^+)}{dt} = \frac{V_L(0^+)}{L} = \frac{V_c(0^+)}{L} = \frac{50}{25\ 10^{-3}}$$

$= 2000$

c) From the solution of Example 8.8, we know that the current response is critically damped. Thus

$$i_L(t) = I_f + D_1' te^{-\alpha t} + D_2' e^{-\alpha t},$$

where

$$\alpha = \frac{1}{2RC} = 40{,}000 \text{ rad/s} \quad \text{and} \quad I_f = 24 \text{ mA}.$$

$$i_L(0) = I_f + D_2' = 29 \text{ mA},$$

from which we get

$$D_2' = 29 - 24 = 5 \text{ mA}.$$

The solution for D_1' is

$$\frac{di_L}{dt}(0^+) = D_1' - \alpha D_2' = 2000,$$

or

$$D_1' = 2000 + \alpha D_2'$$
$$= 2000 + (40{,}000)(5 \times 10^{-3})$$
$$= 2200 \text{ A/s} = 2.2 \times 10^6 \text{ mA/s}.$$

Thus the numerical expression for $i_L(t)$ is

$$i_L(t) = (24 + 2.2 \times 10^6 te^{-40{,}000t}$$
$$+ 5e^{-40{,}000t}) \text{ mA}, \quad t \geq 0.$$

d) We can get the expression for $v(t)$, $t \geq 0$ by using the relationship between the voltage and current in an inductor:

$$v(t) = L\frac{di_L}{dt}$$
$$= (25 \times 10^{-3})[(2.2 \times 10^6)(-40{,}000)te^{-40{,}000t}$$
$$+ 2.2 \times 10^6 e^{-40{,}000t}$$
$$+ (5)(-40{,}000)e^{-40{,}000t}] \times 10^{-3}$$
$$= -2.2 \times 10^6 te^{-40{,}000t} + 50e^{-40{,}000t} \text{ V}, t \geq 0.$$

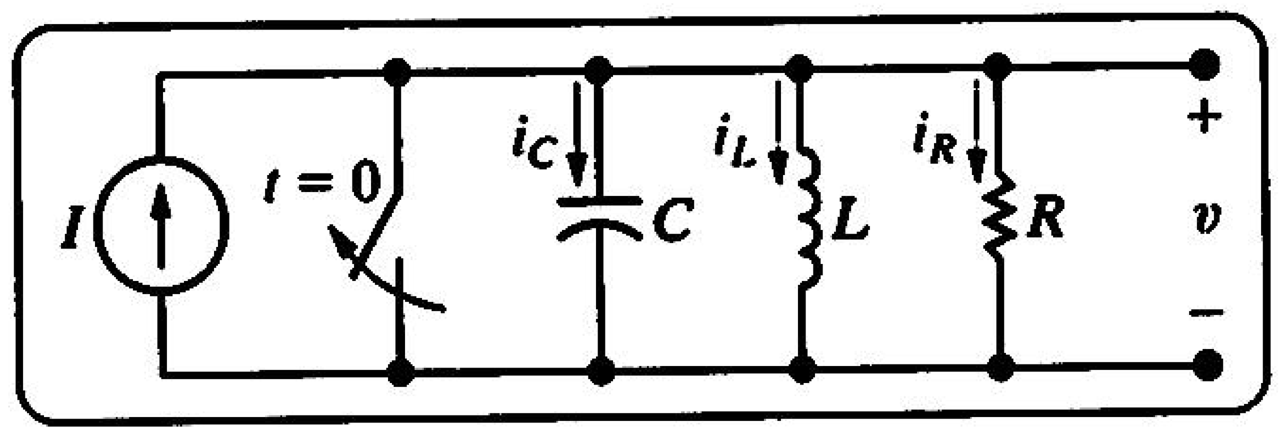
To check this result, let's verify that the initial voltage across the inductor is 50 V:

$$v(0) = -2.2 \times 10^6(0)(1) + 50(1) = 50 \text{ V}.$$

ASSESSING OBJECTIVE 1

◆ **Be able to determine the natural response and the step response of parallel *RLC* circuits**

8.6 In the circuit shown, $R = 500\ \Omega$, $L = 0.64$ H, $C = 1\ \mu$F, $I_0 = 0.5$ A, $V_0 = 40$ V, and $I = -1$ A. Find (a) $i_R(0^+)$; (b) $i_C(0^+)$; (c) $di_L(0^+)/dt$; (d) s_1, s_2; (e) $i_L(t)$ for $t \geq 0$; and (f) $v(t)$ for $t \geq 0^+$.



ANSWER: (a) 80 mA; (b) −1.58 A; (c) 62.5 A/s; (d) $(-1000 + j750)$ rad/s, $(-1000 - j750)$ rad/s; (e) $[-1 + e^{-1000t}[1.5\cos 750t + 2.0833\sin 750t]$ A, for $t \geq 0$; (f) $e^{-1000t}(40\cos 750t - 2053.33\sin 750t)$ V, for $t \geq 0^+$.

Solution: $V_c = V_L = V_R \Rightarrow I_R(0^+) = \frac{V_c(0^+)}{R} = \frac{40}{500} = 0.08A = 80mA$

b) $I = I_c + I_L + I_R$

$-1 = I_c(0^+) + 0.5 + 0.08 \Rightarrow I_c(0^+) = -1.58\ A$

c) $\frac{di_L}{dt} = \frac{V_L}{L}$ $\quad \frac{di_L(0^+)}{dt} = \frac{V_c(0^+)}{L} = \frac{40}{0.64} = 62.5\ \frac{A}{s}$

d) circuit equation

$\frac{di_L^2}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{1}{LC} i_L = 0$ $\quad s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$ $\quad s^2 + 2000s + 1.562500 = 0$

$s^2 + 2000s + 1562500 = 0$ $\quad s_1 = \frac{-2000 + \sqrt{2000^2 - 4 \times 1562500}}{2} = -1000 + j750$

$s_2 = -1000 - j750$

Or using the formula

$\alpha = \frac{1}{2RC} = 1000$ $\quad \omega_0^2 = \frac{1}{LC} = 1562500$

$\omega d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{1562500 - 1000^2} = 750$

$s_1 = -\alpha + j\omega d = -1000 + j750$

$s_2 = -\alpha - j\omega d = -1000 - j750$

$$i_L(t) = I + e^{-\alpha t}\left(B_1 \cos w_d t + B_2 \sin w_d t\right)$$

$$\alpha = 1000 \qquad w_d = 750$$

$$i_L(0) = 0.5\,A = -1 + e^{0}\left(B_1 \cos 0 + B_2 \sin 0\right)$$

$$0,5 + 1 = B_1 \quad \therefore \quad \boxed{B_1 = 1.5}$$

$$\frac{di_L}{dt} = 0 + (-\alpha)e^{\alpha t}\left(B_1 \cos w_d t + B_2 \sin w_d t\right)$$
$$+ e^{-\alpha t}\left(B_1(-w_d) \sin w_d t + B_2 w_d \cos w_d t\right)$$

$$62,5 = -\alpha\, e^{0}(B_1 + 0) + e^{0}(0 + B_2 w_d)$$

$$B_2 = \frac{1}{w_d}\left(62,5 + \alpha B_1\right) = \frac{1}{750}\left(62,5 + 10000 \times 1.5\right) = 2,083$$

$$\boxed{i_L(t) = -1 + e^{-1000t}\left(1,5 \cos 750t + 2,08 \sin 750t\right)}$$

from above

$$\frac{di_L}{dt} = e^{\alpha t}\left[\cos w_d t\,\underbrace{\left(-\alpha B_1 + w_d B_2\right)}_{62,25} + \sin w_d t\,\underbrace{\left(-\alpha B_2 - w_d B_1\right)}_{3205}\right]$$

$$V = L\frac{di_L}{dt} = 0.64\left(\qquad\qquad\right)$$

$$\boxed{V_L(t) = e^{-1000t}\left(40 \cos 750t - 2053 \sin 750t\right)}$$

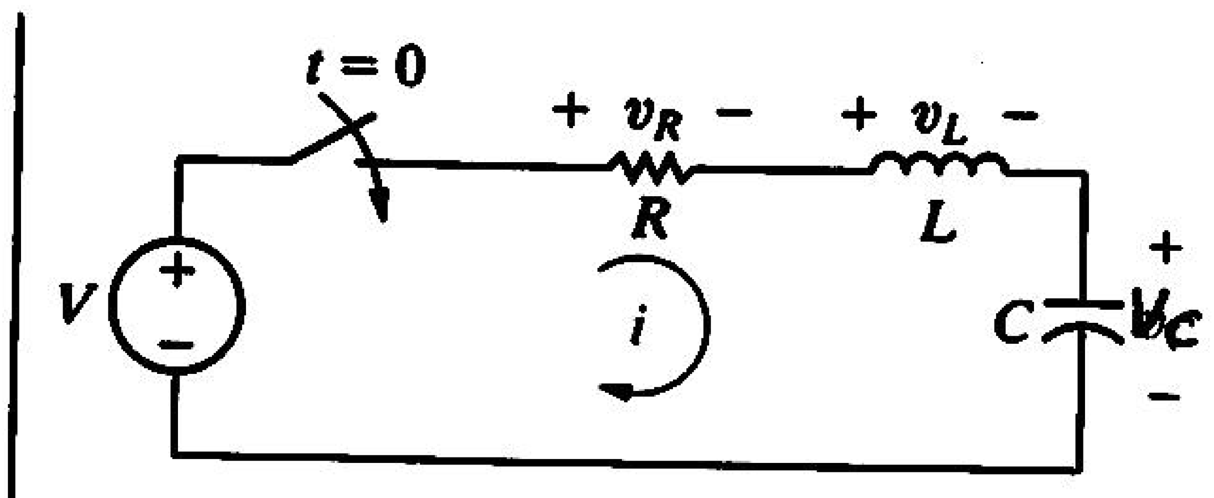


Figure 8.15 A circuit used to illustrate the step response of a series *RLC* circuit.

$$-V + Ri + V_L + V_c = 0$$

$$-V + Ri + L\frac{di_L}{dt} + V_c = 0$$

take derivative of both sides

V : constant $\Rightarrow$ $\frac{dV}{dt} = 0$

$$R\frac{di_L}{dt} + L\frac{d^2 i_L}{dt^2} + \frac{dV_c}{dt} = 0$$

$i_c = C\frac{dV_c}{dt}$ $\quad \frac{dV_c}{dt} = \frac{1}{C} i_c = \frac{1}{C} i_L$ $\quad (i_c = i_L = i_R)$

$$R\frac{di}{dt} + L\frac{d^2 i}{dt^2} + \frac{1}{C} i = 0$$

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC} i = 0$$

$\boxed{s^2 + \frac{R}{L}s + \frac{1}{LC} = 0}$ characteristic equation

$$-V + Ri + V_L + V_c = 0$$

$$-V + Ri + L\frac{di}{dt} + V_c$$

$$i = C\frac{dV_c}{dt} \qquad \frac{di}{dt} = C\frac{d^2V_c}{dt^2}$$

$$-V + RC\frac{dV_c}{dt} + LC\frac{d^2V_c}{dt^2} + V_c = 0$$

$$\frac{d^2V_c}{dt^2} + \frac{R}{L}\frac{dV_c}{dt} + \frac{1}{LC}V_c = \frac{1}{LC}V$$

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$ (Characteristic equation)

$$s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}},$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}.$$

$$\alpha = \frac{R}{2L} \text{ rad/s,} \qquad \omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s.}$$

$$v_C = V_f + A_1'e^{s_1t} + A_2'e^{s_2t} \text{ (overdamped),} \quad (8.67)$$

$$v_C = V_f + B_1'e^{-\alpha t}\cos\omega_d t + B_2'e^{-\alpha t}\sin\omega_d t \text{ (underdamped),} \quad (8.68)$$

$$v_C = V_f + D_1'te^{-\alpha t} + D_2'e^{-\alpha t} \text{ (critically damped),} \quad (8.69)$$

$$V_f = V_c(\infty) = V$$

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EXAMPLE 8.11

The 0.1 μF capacitor in the circuit shown in Fig. 8.16 is charged to 100 V. At $t = 0$ the capacitor is discharged through a series combination of a 100 mH inductor and a 560 Ω resistor.

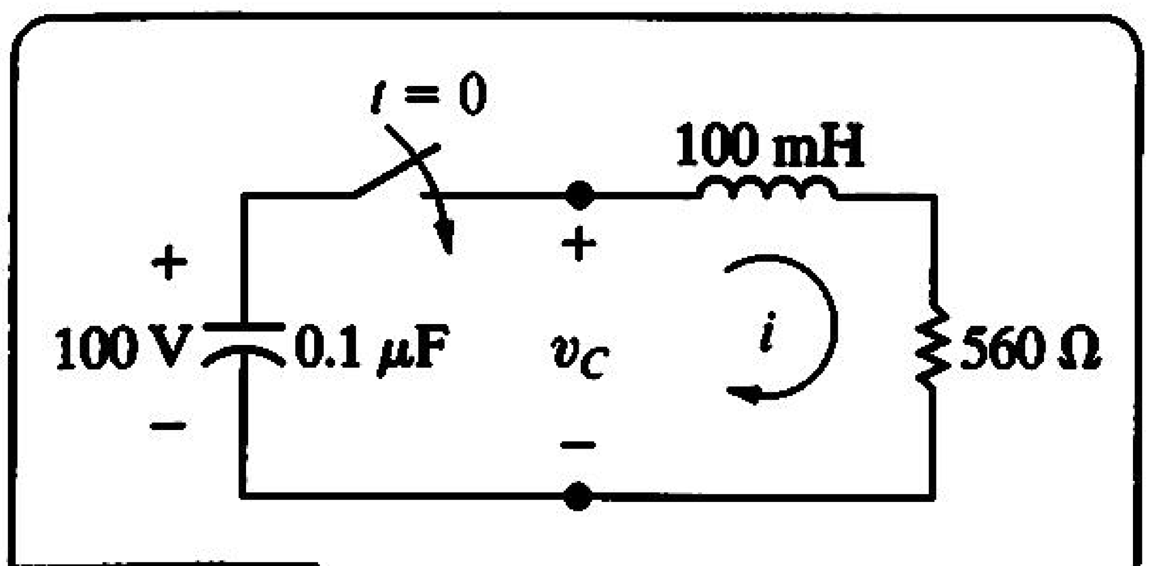

Figure 8.16 The circuit for Example 8.11.

a) Find $i(t)$ for $t \geq 0$.

b) Find $v_C(t)$ for $t \geq 0$.

SOLUTION

$$\frac{dv^2}{dt^2} + \frac{R}{L}\frac{dV_c}{dt} + \frac{1}{LC}V_c = 0$$

$$s^2 + \frac{R}{L}s + \frac{1}{LC}V = 0$$

$$s^2 + \frac{560}{0.1}s + \frac{1}{0.1 \times 0.1\ 10^{-6}} = 0$$

$s_1 = -2800 + j\,9600$

$s_2 = -2800 - j\,9600$

or

$$\alpha = \frac{R}{2L} = \frac{560}{2(100)} \times 10^3 = 2800 \text{ rad/s.}$$

$$\omega_0^2 = \frac{1}{LC} = \frac{(10^3)(10^6)}{(100)(0.1)} = 10^8,$$

$$\alpha^2 = 7.84 \times 10^6 = 0.0784 \times 10^8.$$

$\alpha^2 < \omega_0^2$ complex roots

$$s_{12} = -\alpha \mp j\sqrt{\omega_0^2 - \alpha^2}$$

$$= -2800 \mp j\,9600$$

$$i(t) = B_1 e^{-\alpha t}\cos\omega_d t + B_2 e^{-\alpha t}\sin\omega_d t,$$

$$i(0) = 0 = B_1.$$

$$L\frac{di(0^+)}{dt} = V_0,$$

or

$$\frac{di(0^+)}{dt} = \frac{V_0}{L} = \frac{100}{100} \times 10^3$$

$$= 1000 \text{ A/s.}$$

Because $B_1 = 0$,

$$i(t) = B_2 e^{-at}\sin W_d t$$

$$\frac{di}{dt} = 400B_2e^{-2800t}(24\cos 9600t - 7\sin 9600t).$$

Thus

$$\frac{di(0^+)}{dt} = 9600B_2,$$

$$B_2 = \frac{1000}{9600} \approx 0.1042 \text{ A.}$$

The solution for $i(t)$ is

$$i(t) = 0.1042e^{-2800t}\sin 9600t \text{ A}, \quad t \geq 0.$$

Loop equation

$-V_c + V_L + Ri = 0$

$$V_c = V_L + Ri = L\frac{di}{dt} + Ri$$

$$v_C(t) = (100\cos 9600t + 29.17\sin 9600t)e^{-2800t} \text{ V},$$

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EXAMPLE 8.12

No energy is stored in the 100 mH inductor or the 0.4 μF capacitor when the switch in the circuit shown in Fig. 8.17 is closed. Find $v_C(t)$ for $t \geq 0$.

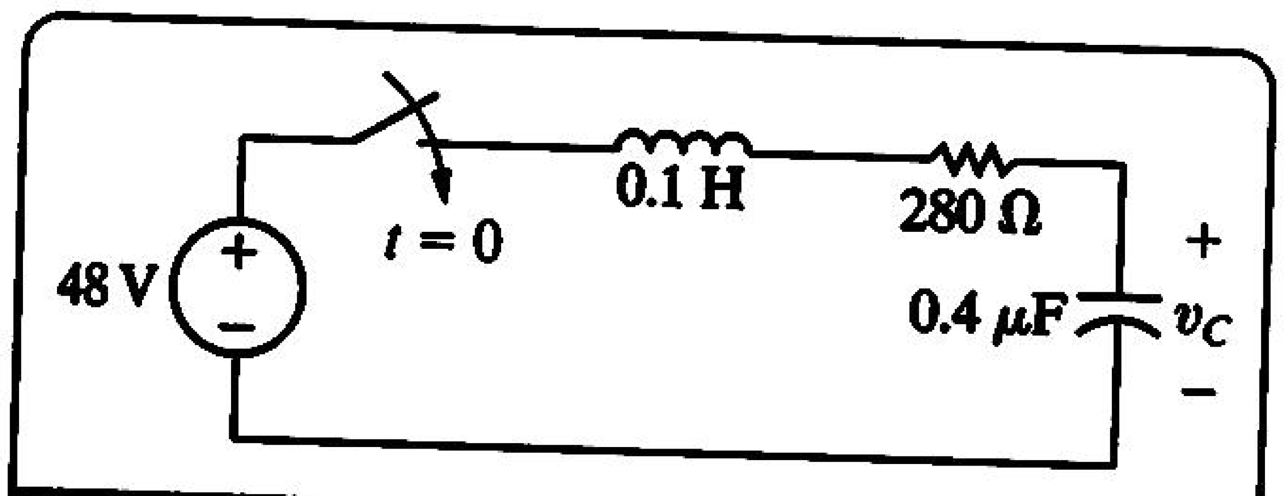

Figure 8.17 The circuit for Example 8.12.

SOLUTION

$$\frac{dv_c^2}{dt^2} + \frac{R}{L}\frac{dV_c}{dt} + \frac{1}{LC}V_c = 48$$

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

The roots of the characteristic equation are

$$s_1 = -\frac{280}{0.2} + \sqrt{\left(\frac{280}{0.2}\right)^2 - \frac{10^6}{(0.1)(0.4)}}$$
$$= (-1400 + j4800) \text{ rad/s},$$
$$s_2 = (-1400 - j4800) \text{ rad/s}.$$

The roots are complex, so the voltage response is underdamped. Thus

$$v_C(t) = 48 + B_1' e^{-1400t} \cos 4800t + B_2' e^{-1400t} \sin 4800t, \quad t \geq 0.$$

No energy is stored in the circuit initially,

$$i_L(0^+) = C\frac{dV_c(0^+)}{dt} = 0$$

$$V_c(0^+) = 0$$

$$v_C(0) = 0 = 48 + B_1'$$

$$\frac{dv_C(0^+)}{dt} = 0 = 4800B_2' - 1400B_1'.$$

Solving for B_1' and B_2' yields

$$B_1' = -48 \text{ V},$$
$$B_2' = -14 \text{ V}.$$

Therefore, the solution for $v_C(t)$ is

$$v_C(t) = (48 - 48e^{-1400t} \cos 4800t - 14e^{-1400t} \sin 4800t) \text{ V}, \quad t \geq 0.$$