

12.1 ♦ Definition of the Laplace Transform

The Laplace transform of a function is given by the expression

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st} dt, \quad (12.1)$$

where the symbol $\mathcal{L}\{f(t)\}$ is read "the Laplace transform of $f(t)$."

The Laplace transform of $f(t)$ is also denoted $F(s)$; that is,

$$F(s) = \mathcal{L}\{f(t)\}. \quad (12.2)$$

12.2 ♦ The Step Function

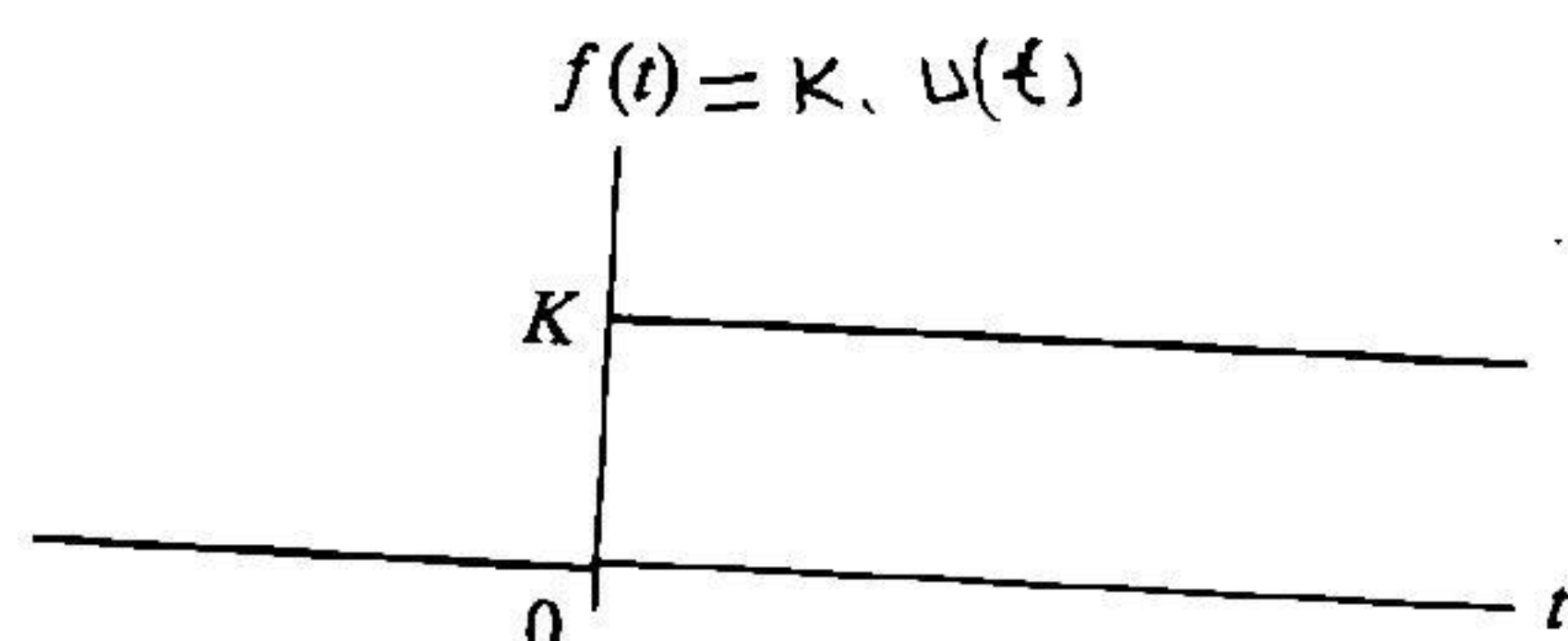
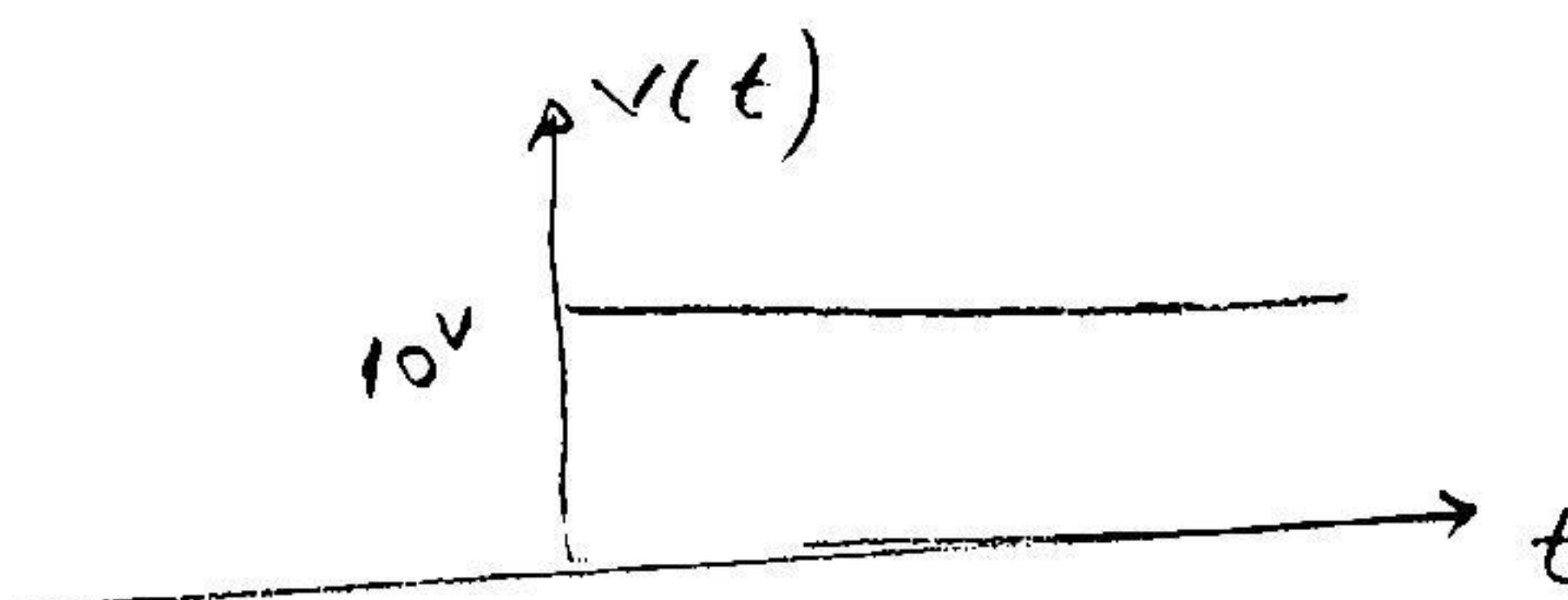
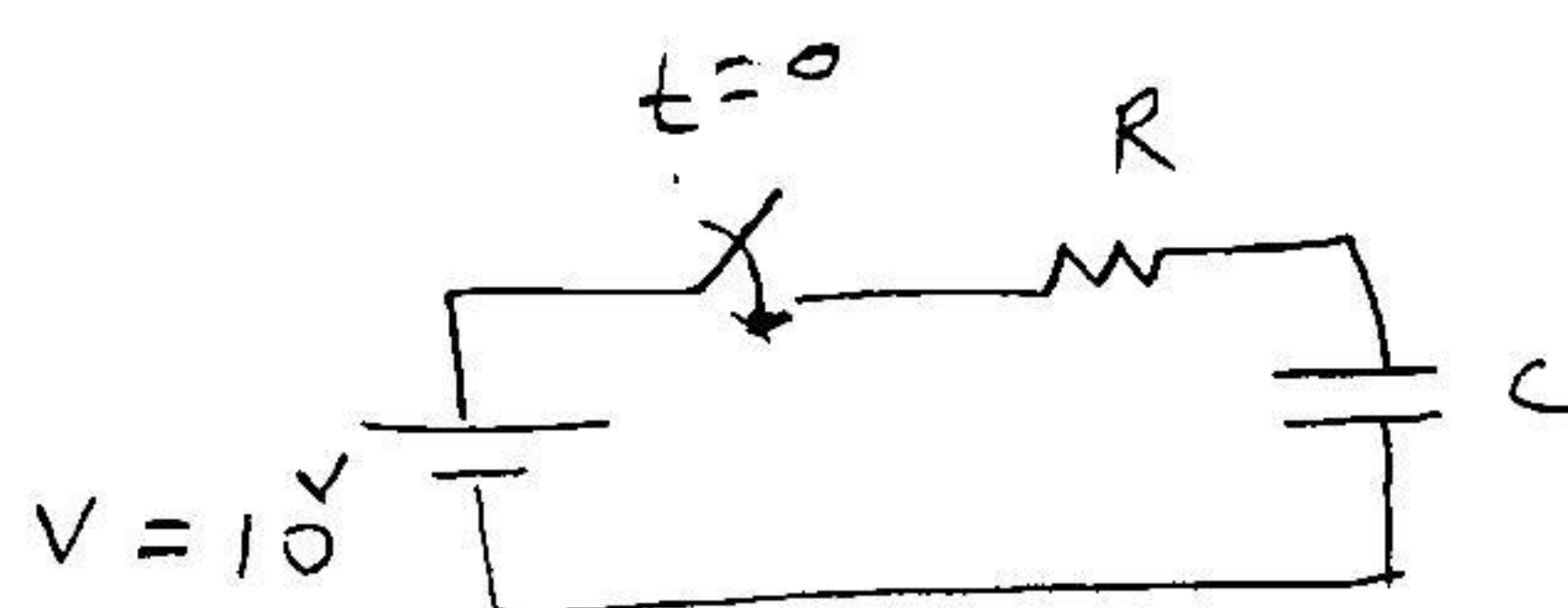


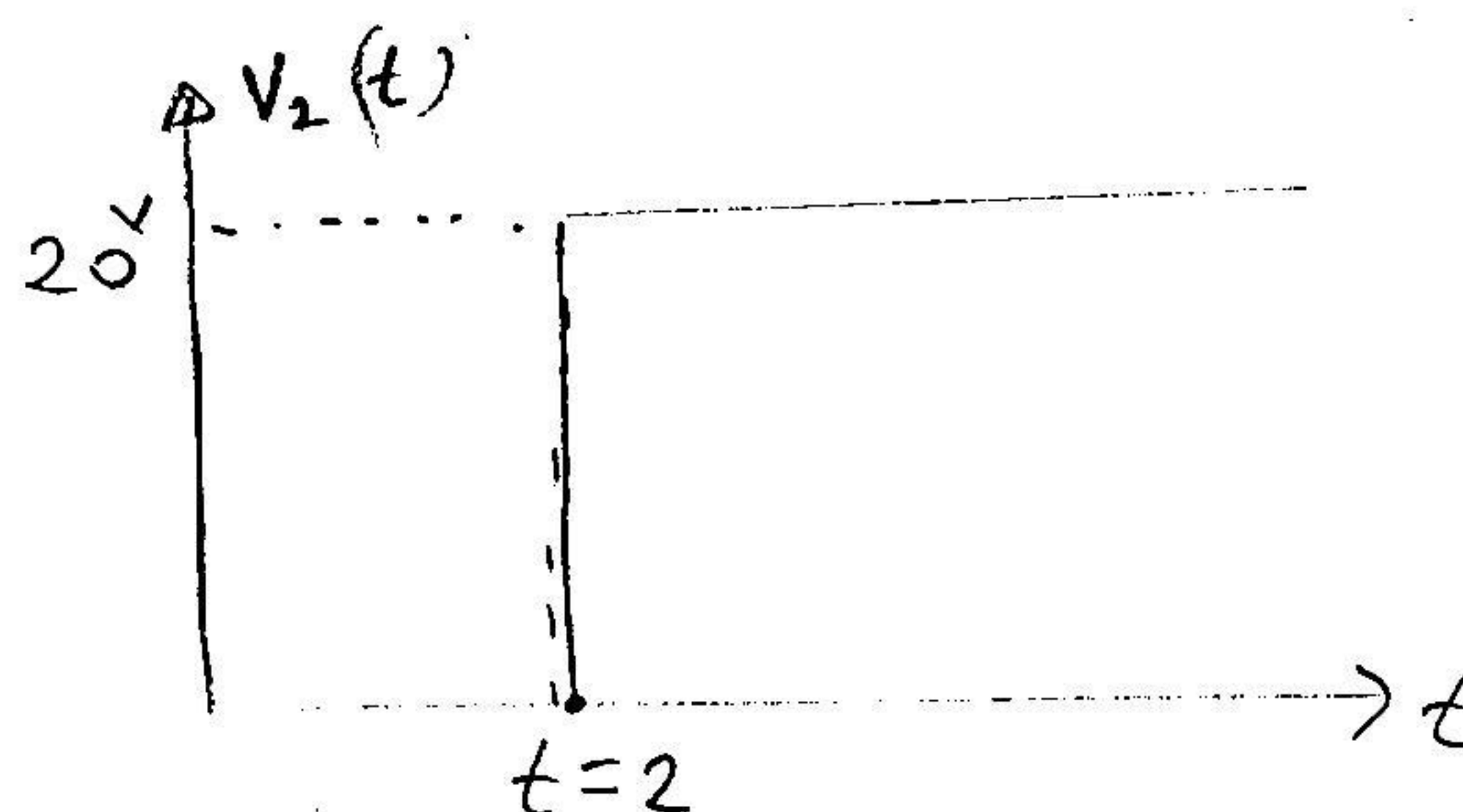
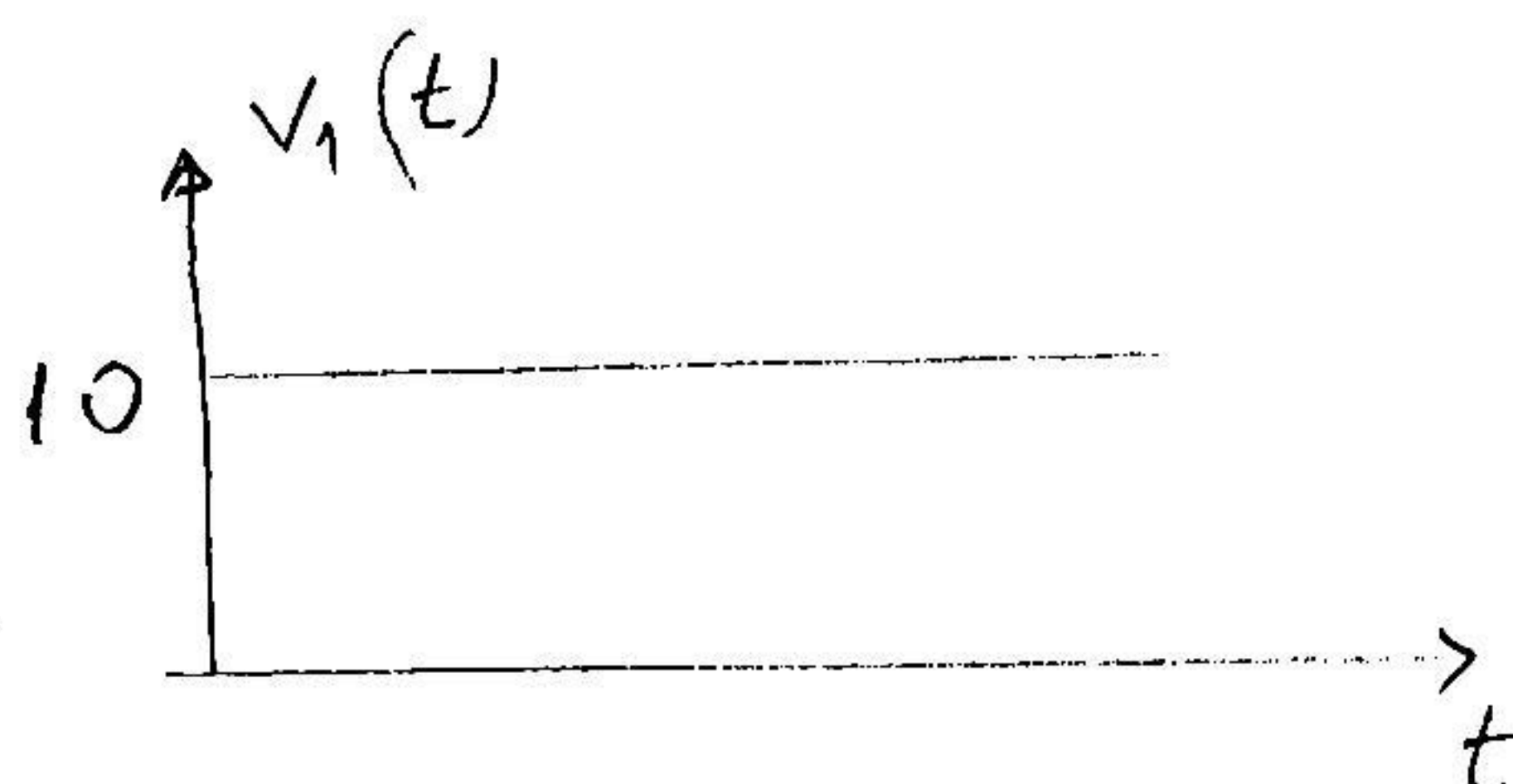
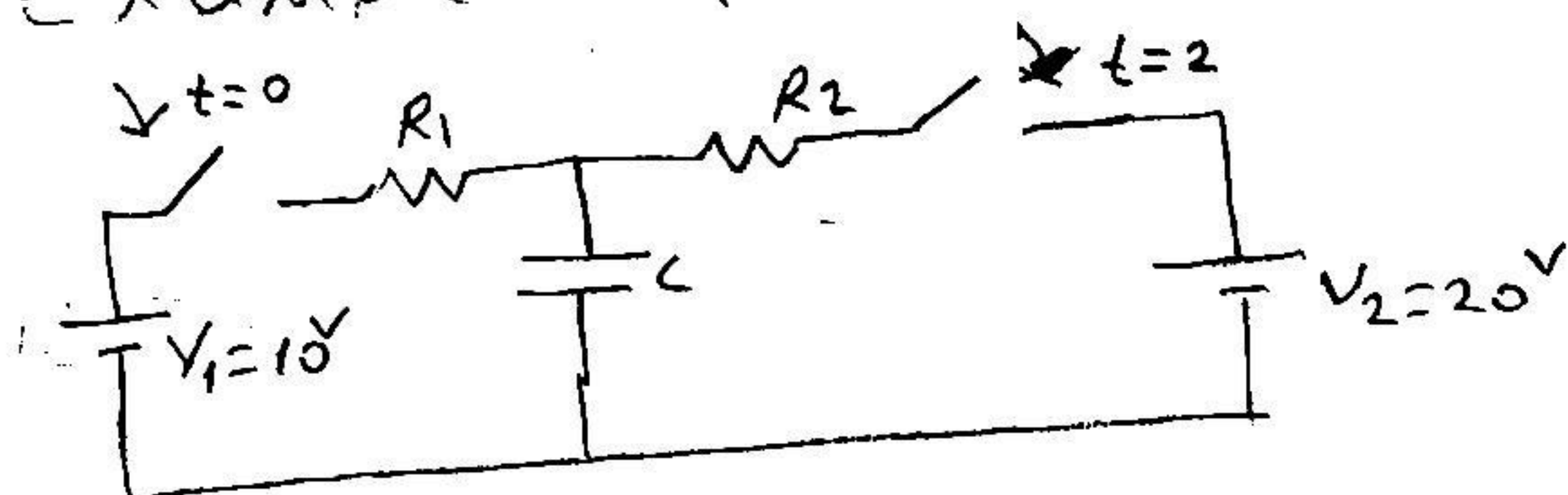
Figure 12.2 The step function.



$$V(t) = 0 \quad \text{for } t < 0$$

$$V(t) = 10^V \quad \text{for } t \geq 0$$

Example problem: at $t=0$ switch 1 is closed, at $t=2$ switch 2 is closed. Draw $V_1(t)$, $V_2(t)$



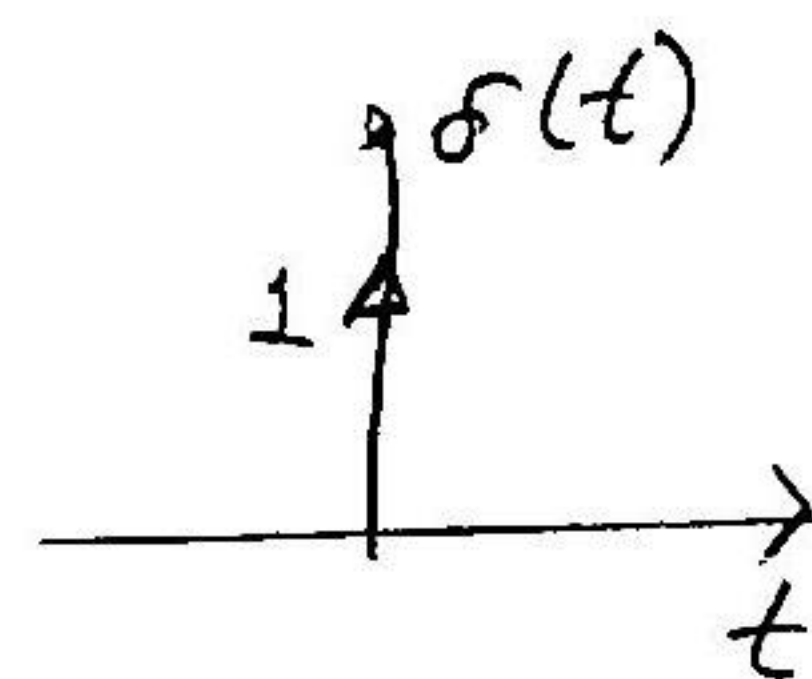
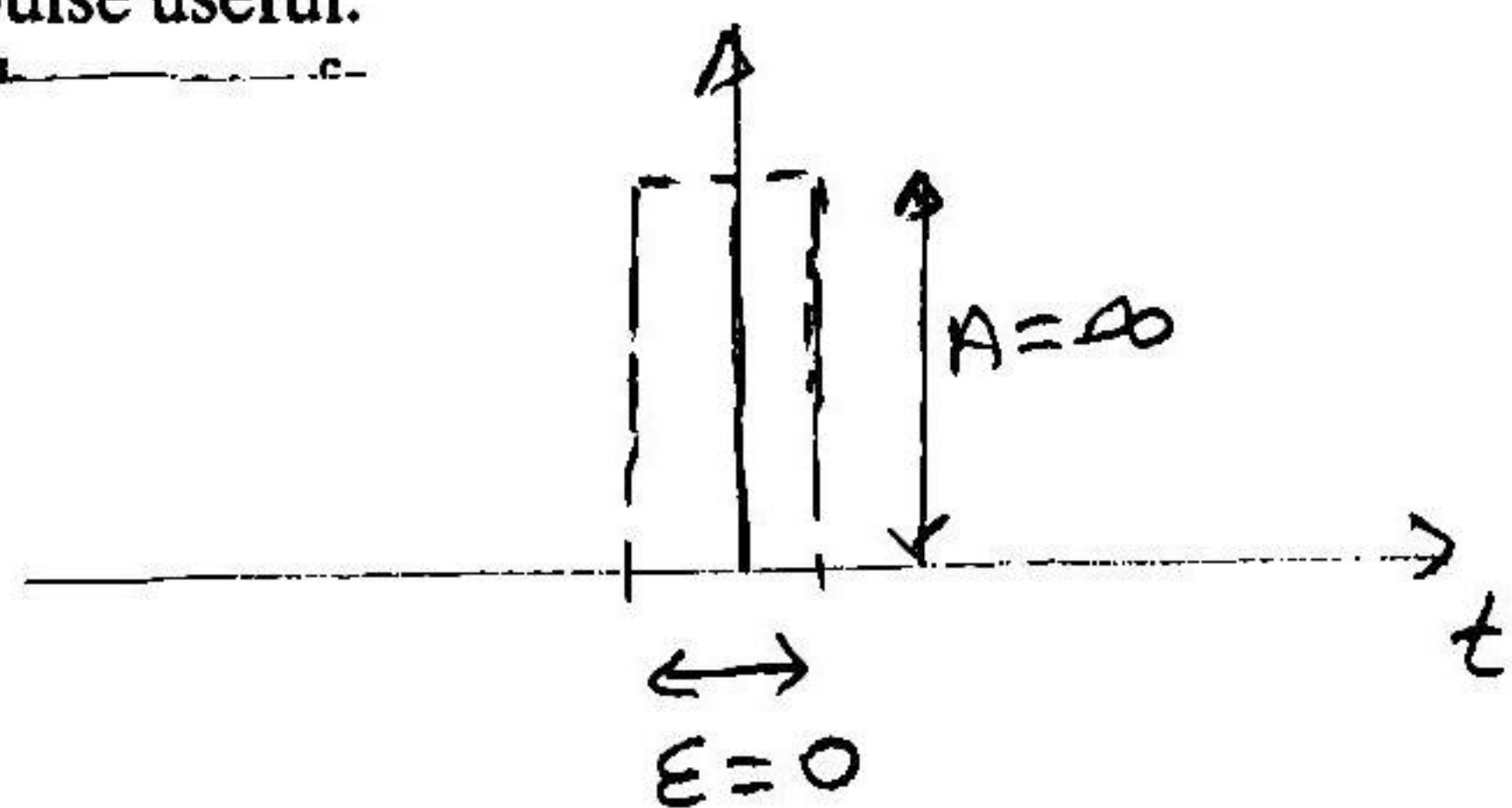
$$V_2(t) = 0 \quad \text{for } t < 2$$

$$V_2(t) = 20 \quad \text{for } t \geq 2$$

12.3 ♦ The Impulse Function

312

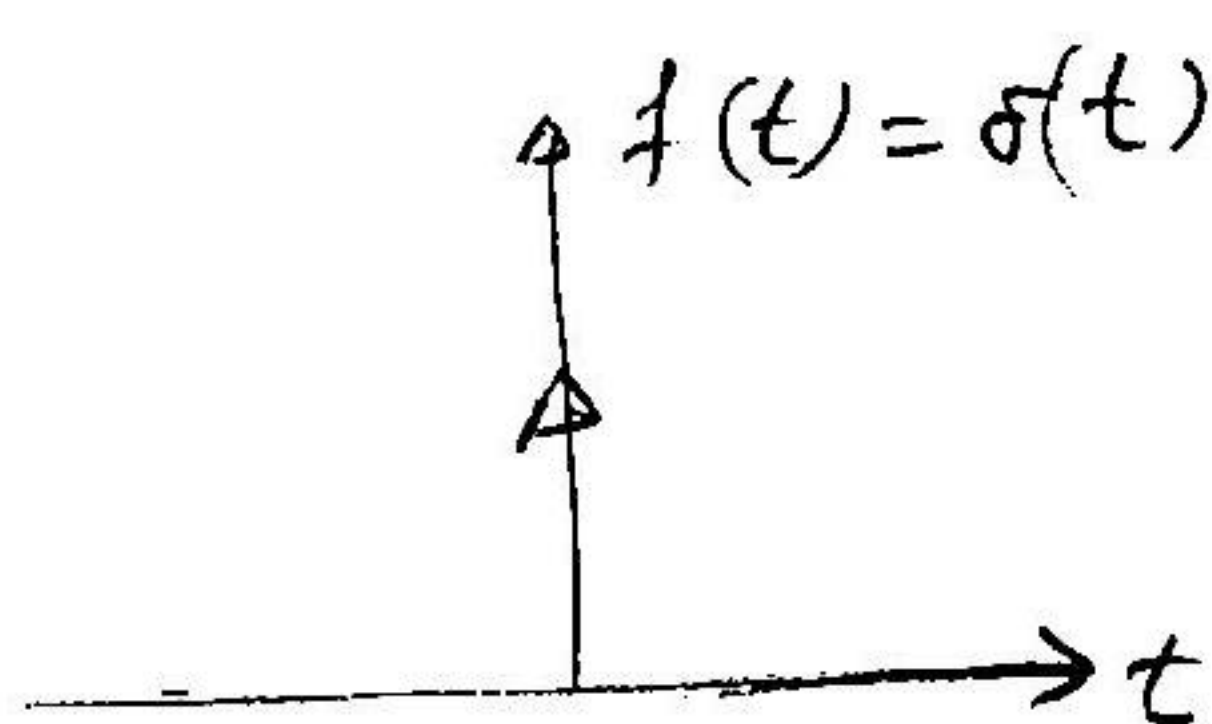
An **impulse** is a signal of infinite amplitude and zero duration. Such signals don't exist in nature, but some circuit signals come very close to approximating this definition, so we find a mathematical model of an impulse useful.



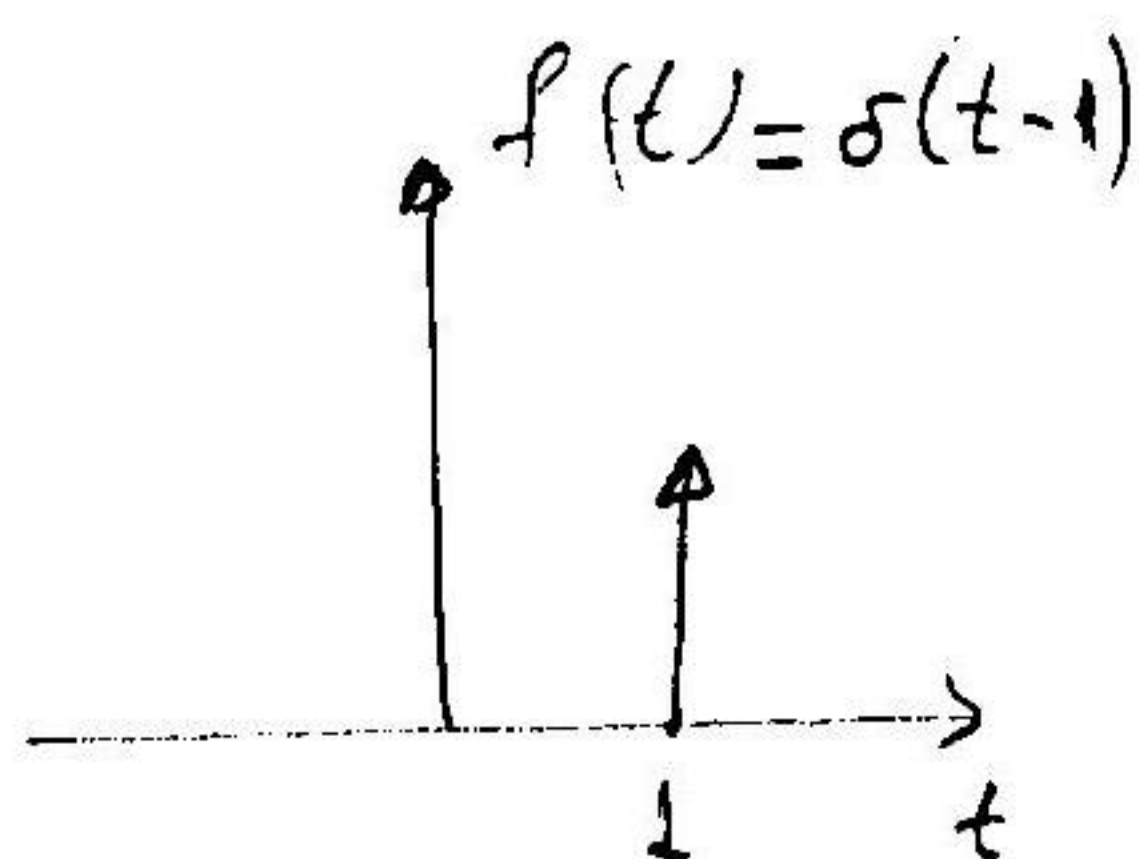
Example problem. Show impulse function at

- a) $t=0$, b) $t=1$ c) $t=10$ d) $t=-2$

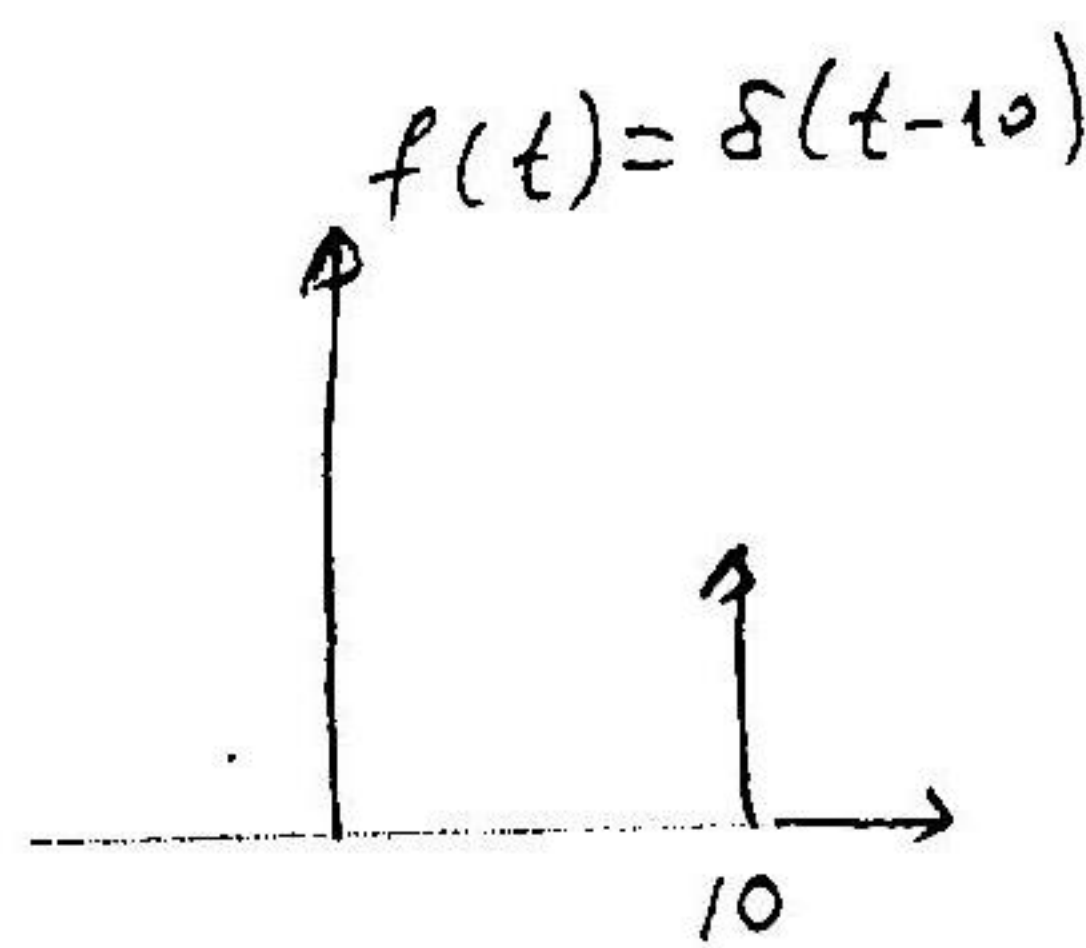
Solution



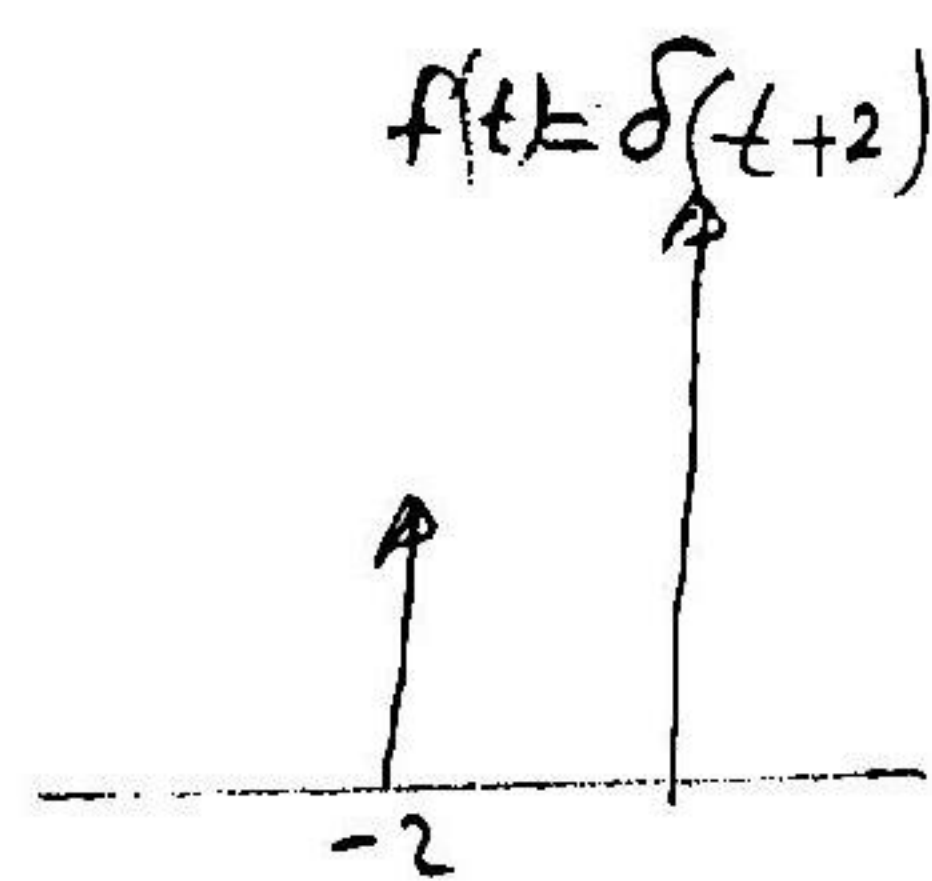
a)



b)

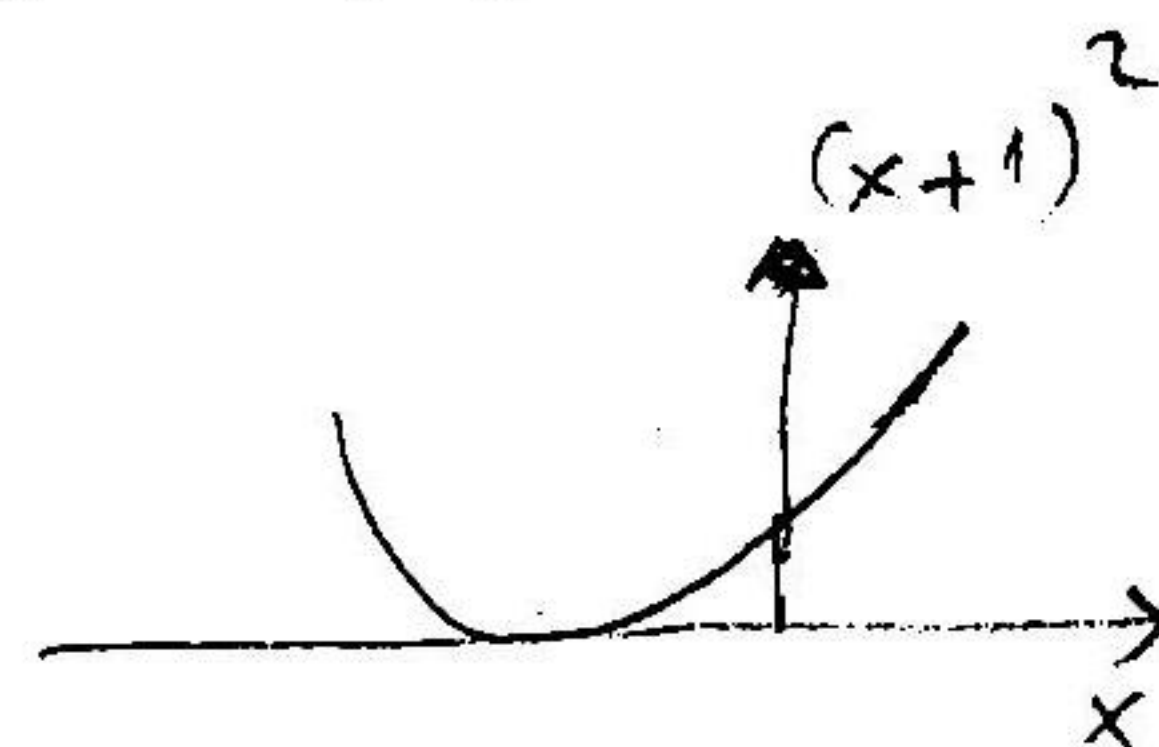
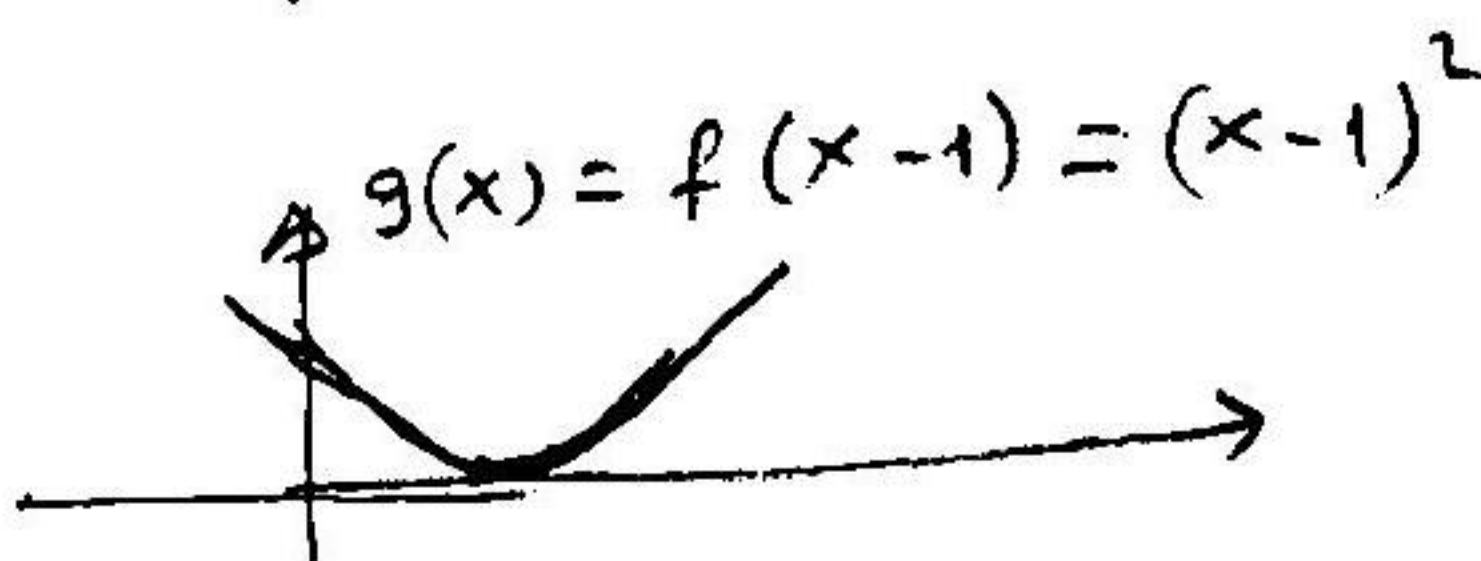
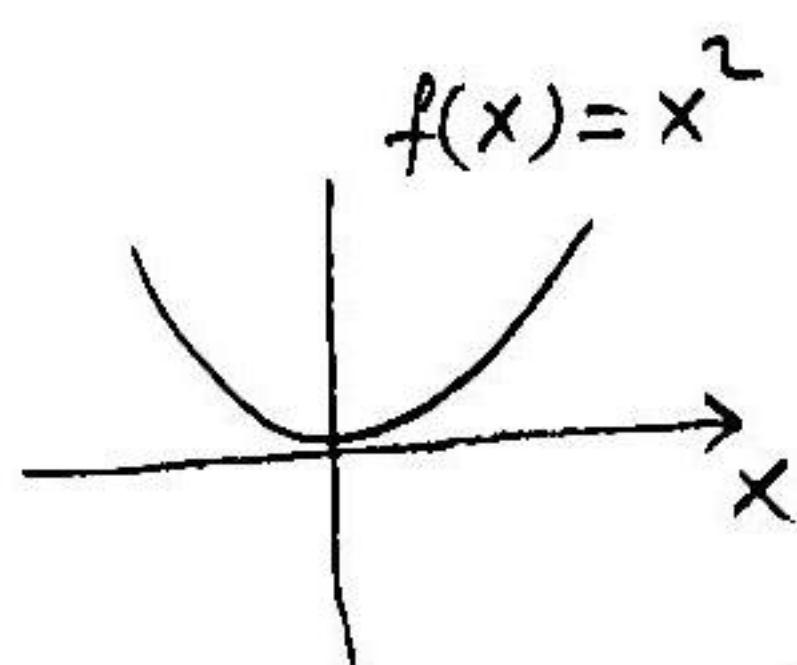
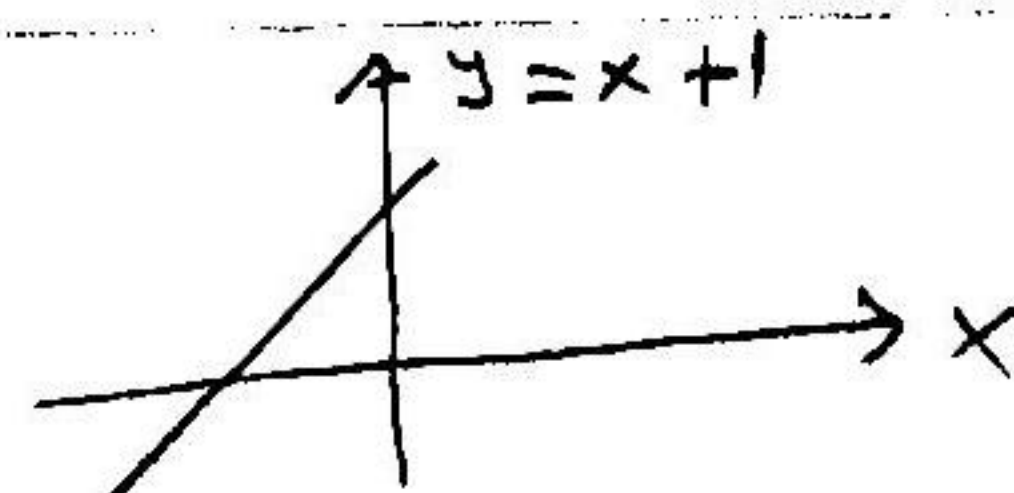
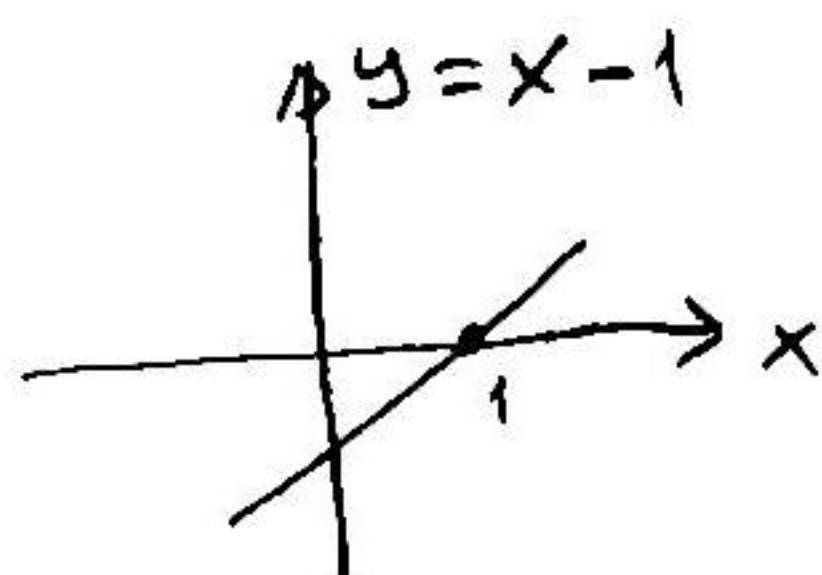
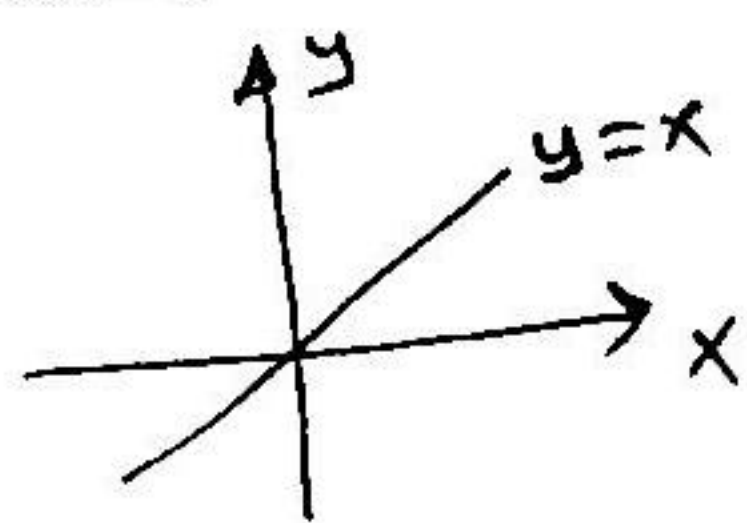


c)



d)

Note:

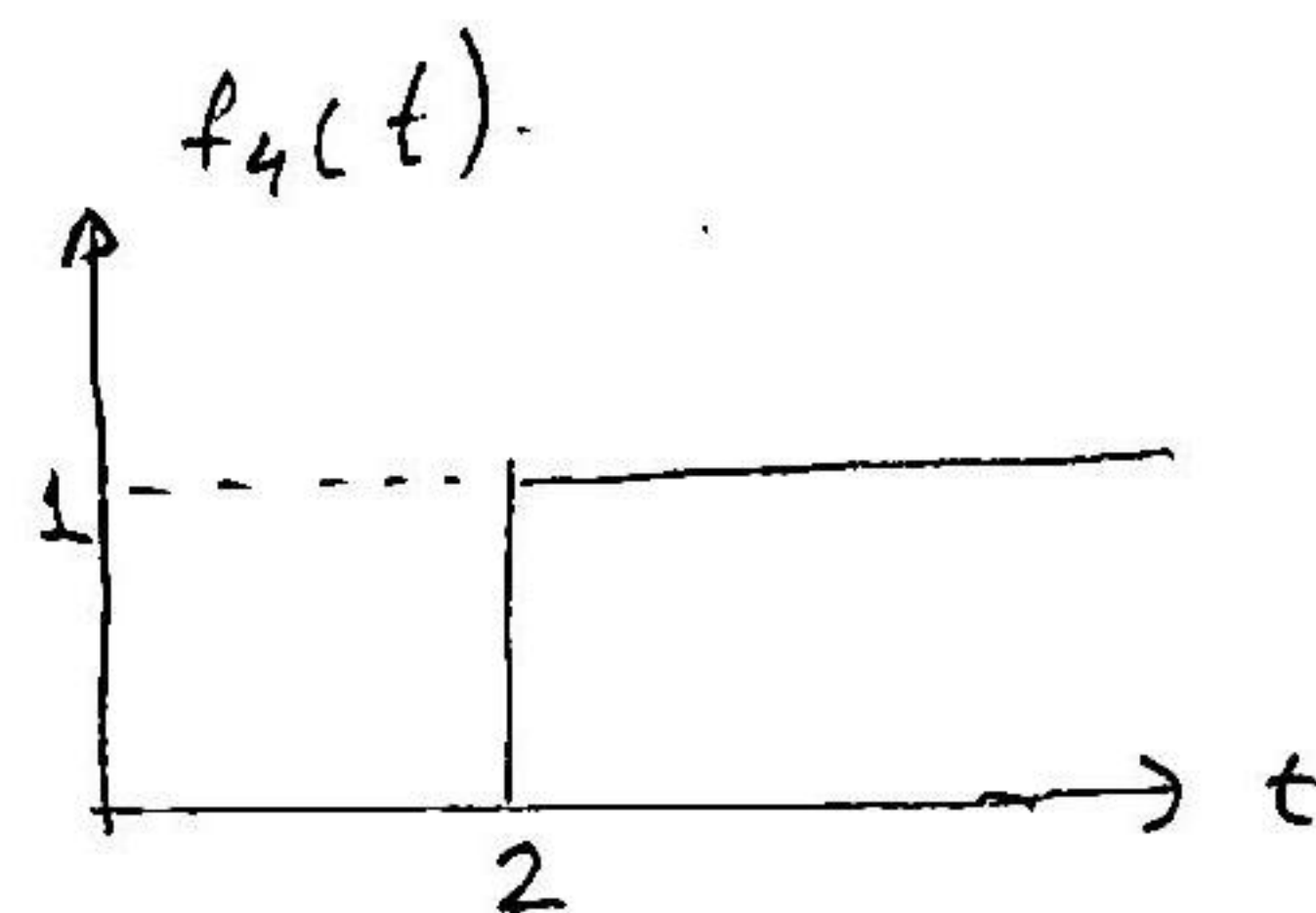
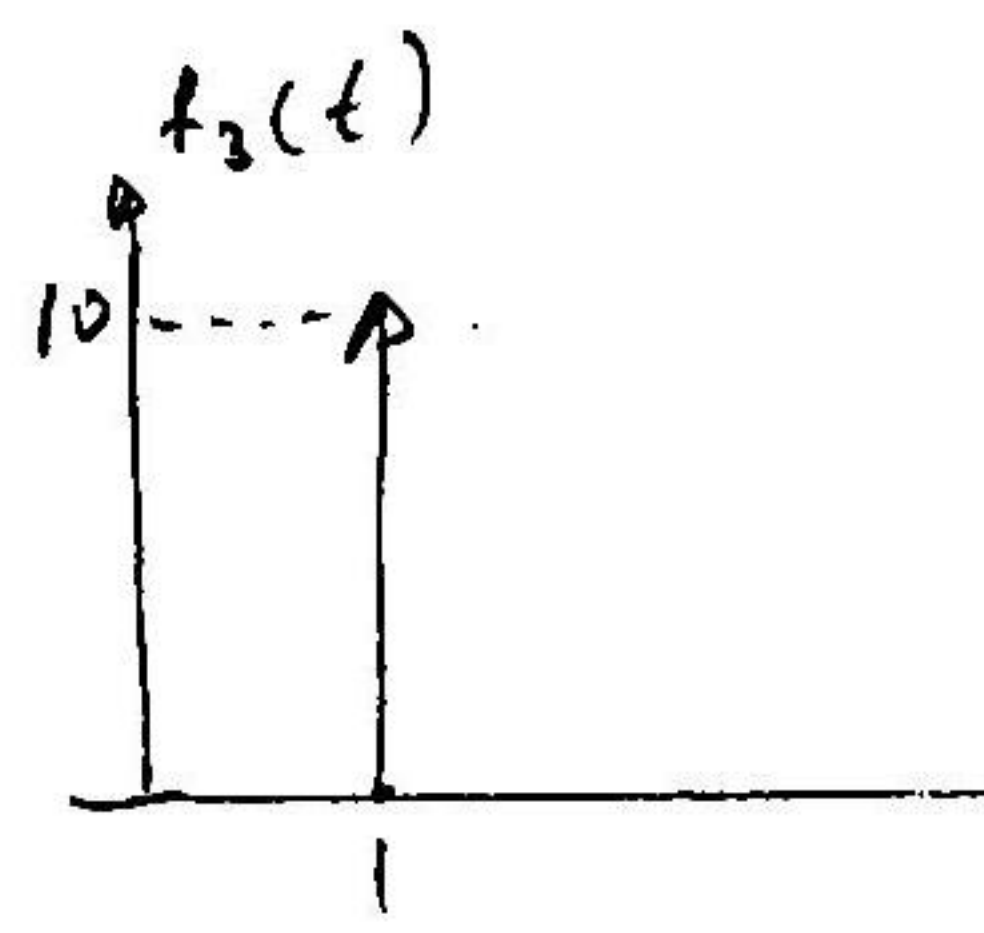
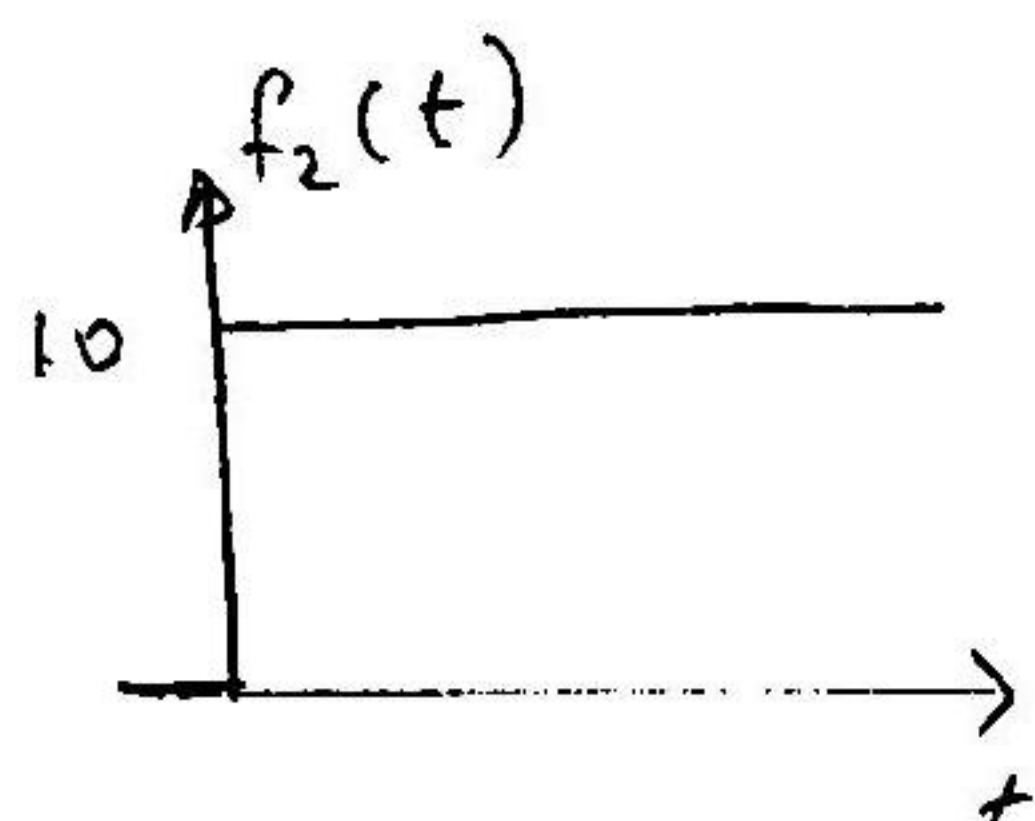
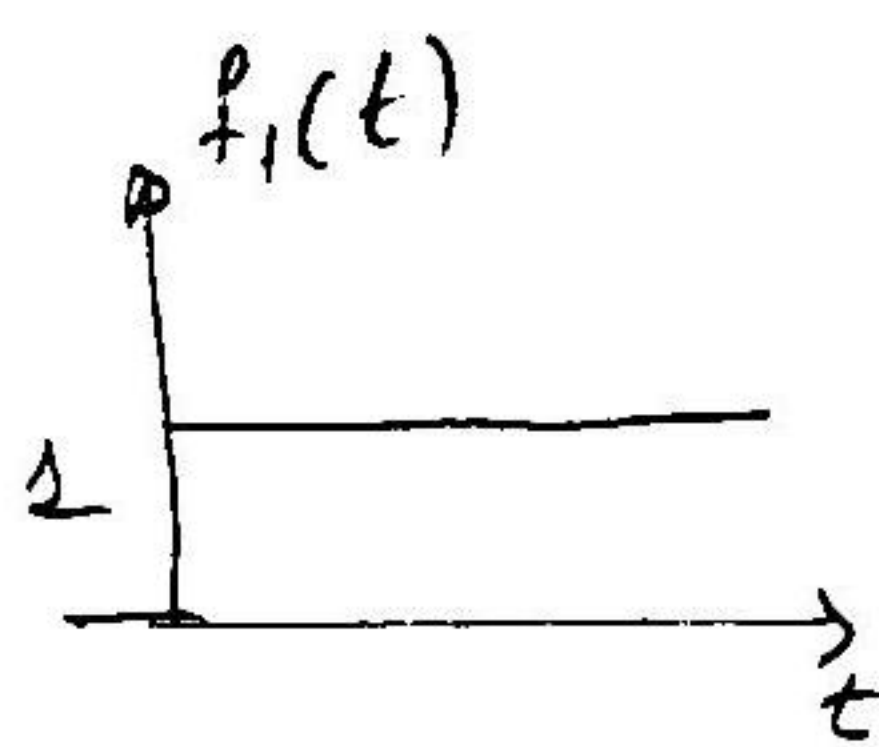


Example problem: Draw the following functions

- a) $f_1(t) = 1$ b) $f_2(t) = 10 \mathcal{U}(t)$ c) $f_3(t) = 10\delta(t-1)$

- d) $f_4(t) = \mathcal{U}(t-2)$

Solution



$$\delta(t) = 0 \quad \text{for } t \neq 0$$

$$\delta(t) \neq 0 \quad \text{for } t = 0$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

$$\int_{-\infty}^{\infty} \delta(t-a) dt = 1$$

$$\int_{-\infty}^{\infty} \delta(t) f(t) dt = f(0)$$

$$\int_{-\infty}^{\infty} \delta(t-a) f(t) dt = f(a)$$

Example problem. Calculate the following integrals

$$S_1 = \int_{-\infty}^{\infty} \delta(t) t^2 dt$$

$$S_1 = \left. t^2 \right|_{t=0} = 0^2 = 0$$

$$S_2 = \int_{-\infty}^{\infty} \delta(t-2) (t^2+5) dt$$

$$S_2 = \left. (t^2+5) \right|_{t=2} = 2^2+5 = 9$$

$$S_3 = \int_{-\infty}^{\infty} \delta(t) e^{t^2+1} dt$$

$$S_3 = \left. e^{t^2+1} \right|_{t=0} = e^1 = 2.71$$

$$S_4 = \int_{-\infty}^{\infty} \delta(t-3) \cos(2t+2) dt$$

$$S_4 = \cos(2 \times 3 + 2) = -0.14$$

↑
radian

Laplace Transform

5-3'21

Laplace Transform eases the solution of differential equations.

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

Laplace Transform exists if $f(t)$ is bounded.

$$\lim_{t \rightarrow \infty} f(t) e^{-st} < \infty$$

Example $f(t) = e^{at}$ $F(s) = ?$

$$\begin{aligned} F(s) &= \int_0^{\infty} f(t) e^{-st} dt = \int_0^{\infty} e^{at} e^{-st} dt = \int_0^{\infty} e^{(a-s)t} dt \\ &= \frac{1}{a-s} e^{(a-s)t} \Big|_0^{\infty} = \frac{1}{a-s} (0 - 1) = \frac{-1}{a-s} = \frac{1}{s-a} \end{aligned}$$

" $f(t)$ is bounded" means $e^{(a-s)\infty} \rightarrow \text{not } \infty$

Result $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

$$\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$$

Example calculate $\mathcal{L}\{\delta(t)\}$

$$F(s) = \int_0^{\infty} \delta(t) e^{-st} dt = e^0 = 1$$

Note: definition of dirac function is

$$\int_{-\infty}^{\infty} \delta(t) f(t) dt = f(0)$$

Example 3: $\mathcal{L}\{\cos \omega t\} = ?$

S-322

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathcal{L}\{e^{j\omega t}\} = \frac{1}{s-j\omega}$$

$$\mathcal{L}\{e^{-j\omega t}\} = \frac{1}{s+j\omega}$$

$$\mathcal{L}\{e^{j\omega t} + e^{-j\omega t}\} = \frac{1}{s-j\omega} + \frac{1}{s+j\omega} = \frac{s+j\omega + s-j\omega}{(s-j\omega)(s+j\omega)} = \frac{2s}{s^2+\omega^2}$$

$$\cos \omega t = \frac{1}{2} \{e^{j\omega t} + e^{-j\omega t}\}$$

$$\mathcal{L}\{\cos \omega t\} = \mathcal{L}\left\{\frac{1}{2}(e^{j\omega t} + e^{-j\omega t})\right\} = \frac{1}{2} \frac{2s}{s^2+\omega^2} = \frac{s}{s^2+\omega^2}$$

$$\boxed{\mathcal{L}\{\cos \omega t\} = \frac{s}{s^2+\omega^2}}$$

Example $\mathcal{L}\{\sin \omega t\} = ?$

$$\mathcal{L}\{e^{j\omega t}\} = \frac{1}{s-j\omega}$$

$$\mathcal{L}\{e^{-j\omega t}\} = \frac{1}{s+j\omega}$$

$$\mathcal{L}\{e^{j\omega t} - e^{-j\omega t}\} = \frac{1}{s-j\omega} - \frac{1}{s+j\omega} = \frac{s+j\omega - (s-j\omega)}{(s-j\omega)(s+j\omega)} = \frac{2j\omega}{s^2+\omega^2}$$

$$\sin \omega t = \frac{1}{2j} \{e^{j\omega t} - e^{-j\omega t}\}$$

$$\mathcal{L}\{\sin \omega t\} = \frac{1}{2j} \mathcal{L}\{e^{j\omega t} - e^{-j\omega t}\} = \frac{1}{2j} \frac{2j\omega}{s^2+\omega^2} = \frac{\omega}{s^2+\omega^2}$$

$$\boxed{\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2}}$$

Properties of Laplace Transform

S-323

OPERATION	$f(t)$	$F(s)$
Multiplication by a constant	$Kf(t)$	$KF(s)$
Addition/subtraction	$f_1(t) + f_2(t) - f_3(t) + \dots$	$F_1(s) + F_2(s) - F_3(s) + \dots$
First derivative (time)	$\frac{df(t)}{dt}$	$sF(s) - f(0^-)$
Second derivative (time)	$\frac{d^2 f(t)}{dt^2}$	$s^2 F(s) - sf(0^-) - \frac{df(0^-)}{dt}$
n th derivative (time)	$\frac{d^n f(t)}{dt^n}$	$s^n F(s) - s^{n-1} f(0^-) - s^{n-2} \frac{df(0^-)}{dt} - s^{n-3} \frac{d^2 f(0^-)}{dt^2} - \dots - \frac{d^{n-1} f(0^-)}{dt^{n-1}}$
Time integral	$\int_0^t f(x) dx$	$\frac{F(s)}{s}$
Translation in time	$f(t-a)u(t-a), a > 0$	$e^{-as} F(s)$
Translation in frequency	$e^{-at} f(t)$	$F(s+a)$
Scale changing	$f(at), a > 0$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
First derivative (s)	$tf(t)$	$-\frac{dF(s)}{ds}$
n th derivative (s)	$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$
s integral	$\frac{f(t)}{t}$	$\int_s^\infty F(u) du$

$$x(0^+) = \lim_{s \rightarrow \infty} sX(s)$$

$$\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} sX(s)$$

$$\mathcal{L}\{x(t-t_0)\} = e^{-st_0} X(s)$$

Example problem 21: $f(t) = e^{\alpha t} \cos bt$ $f(s) = ?$

We know that $\mathcal{L}\{\cos bt\} = \frac{s}{s^2 + b^2}$

shifting in s domain $\mathcal{L}\{e^{\alpha t} x(t)\} = X(s - \alpha)$

$$\mathcal{L}\{e^{\alpha t} \cos bt\} = \left. \frac{s}{s^2 + b^2} \right|_{s \rightarrow s - \alpha} = \frac{s - \alpha}{(s - \alpha)^2 + b^2}$$

$$\mathcal{L}\{e^{-\alpha t} \cos bt\} = \frac{s + \alpha}{(s + \alpha)^2 + b^2}$$

$$\mathcal{L}\{e^{-\alpha t} \sin bt\} = \left. \frac{b}{s^2 + b^2} \right|_{s \rightarrow s + \alpha} = \frac{b}{(s + \alpha)^2 + b^2}$$

Example problem: It is given that

3241

$$\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2 + \omega^2}$$

Calculate $\mathcal{L}\{\cos \omega t\}$ by derivative theorem.

Solution

$$\mathcal{L}\left\{\frac{d}{dt} f(t)\right\} = s f(s) - f(0)$$

$$f(t) = \sin \omega t \Rightarrow \frac{df}{dt} = \omega \cos \omega t \Rightarrow \cos \omega t = \frac{1}{\omega} \frac{df}{dt}$$

$$\cos \omega t = \frac{1}{\omega} \frac{df}{dt}$$

$$\mathcal{L}\{\cos \omega t\} = \mathcal{L}\left\{\frac{1}{\omega} \frac{df}{dt}\right\} = \frac{1}{\omega} \mathcal{L}\left\{\frac{df}{dt}\right\}$$

$$= \frac{1}{\omega} [s f(s) - f(0)]$$

$$= \frac{1}{\omega} \left[s \left(\frac{\omega}{s^2 + \omega^2} \right) - \sin \omega \cdot 0 \right]$$

$$= \frac{s}{s^2 + \omega^2}$$

Example problem: $\mathcal{L}\{\cos \omega t\} = \frac{s}{s^2 + \omega^2}$ find $\mathcal{L}\{\sin \omega t\}$

Solution: $f(t) = \cos \omega t \quad \frac{df}{dt} = -\omega \sin \omega t \quad f(s) = \frac{s}{s^2 + \omega^2}$

$$f(0) = \cos 0 = 1$$

$$\sin \omega t = -\frac{1}{\omega} \frac{df}{dt} \Rightarrow \mathcal{L}\{\sin \omega t\} = -\frac{1}{\omega} \mathcal{L}\left\{\frac{df}{dt}\right\}$$

$$\mathcal{L}\{\sin \omega t\} = -\frac{1}{\omega} [s f(s) - f(0)] = -\frac{1}{\omega} \left[s \frac{s}{s^2 + \omega^2} - 1 \right] = -\frac{1}{\omega} \left[\frac{s^2}{s^2 + \omega^2} - 1 \right]$$

$$= -\frac{1}{\omega} \left[\frac{s^2}{s^2 + \omega^2} - \frac{s^2 + \omega^2}{s^2 + \omega^2} \right] = -\frac{1}{\omega} \left[\frac{s^2 - s^2 - \omega^2}{s^2 + \omega^2} \right] = \frac{\omega}{s^2 + \omega^2}$$

Example Problem: It is given that $\mathcal{L}\{e^{-2t}\} = \frac{1}{s+2}$

calculate $\lim_{t \rightarrow \infty} f(t)$ by final value theorem.

Solution $\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} s f(s)$

$$f(t) = e^{-2t} \quad f(s) = \frac{1}{s+2}$$

$$\lim_{s \rightarrow 0} s f(s) = \lim_{s \rightarrow 0} s \frac{1}{s+2} = 0 \times \frac{1}{0+2} = 0$$

Note: $\lim_{t \rightarrow \infty} f(t) = \lim_{t \rightarrow \infty} e^{-2t} = e^{-\infty} = 0$

Example problem: $\mathcal{L}\{1 - e^{-t}\} = \frac{1}{s} + \frac{1}{s+1}$ verify final value theorem.

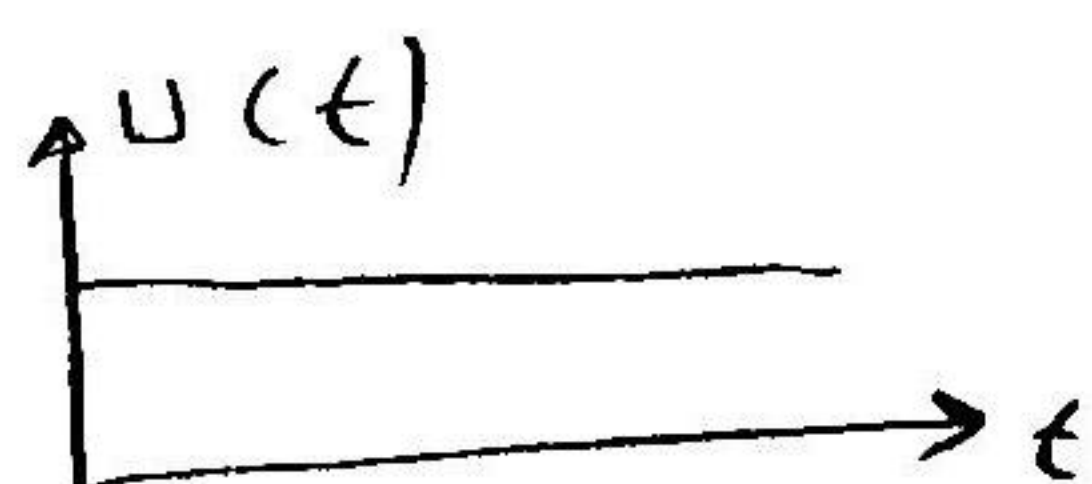
Solution

$$\lim_{t \rightarrow \infty} f(t) = \lim_{t \rightarrow \infty} (1 - e^{-t}) = 1 - e^{-\infty} = 1 - 0 = 1$$

$$\lim_{s \rightarrow 0} s f(s) = \lim_{s \rightarrow 0} s \left(\frac{1}{s} + \frac{1}{s+1} \right) = \lim_{s \rightarrow 0} \left(1 + \frac{s}{s+1} \right) = 1 + \frac{0}{0+1} = 1$$

Example problem: Draw $u(t-2)$ and calculate Laplace Tr.

Solution.



shifting theorem: $\mathcal{L}\{f(t-\alpha)\} = e^{-\alpha s} f(s)$

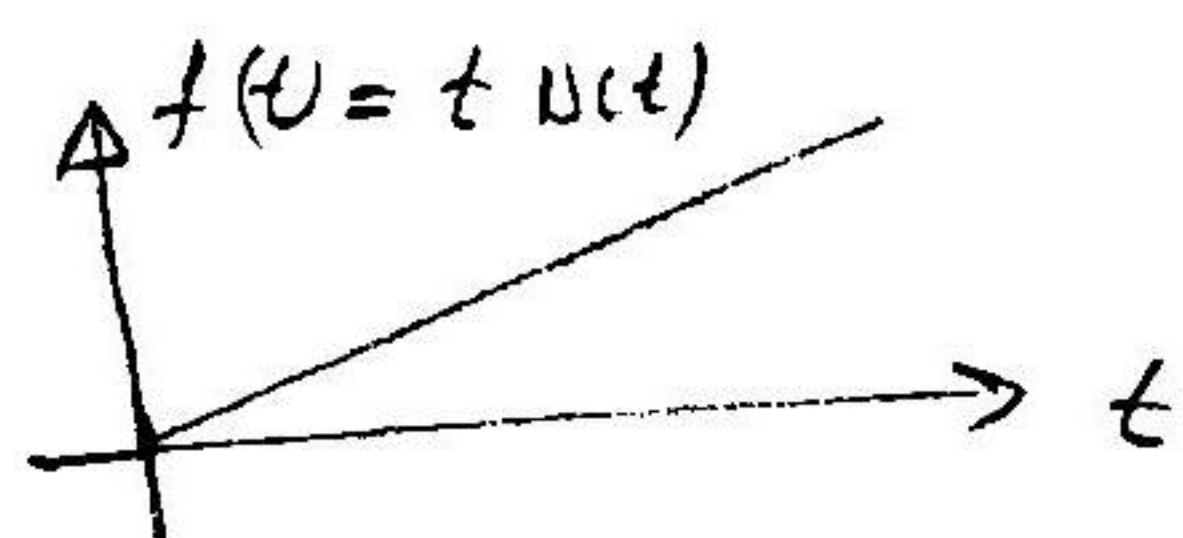
$$\mathcal{L}\{u(t)\} = \frac{1}{s}$$

$$\mathcal{L}\{u(t-2)\} = \frac{1}{s} \times e^{-2s}$$

Example Problem. Calculate $\mathcal{L}\{t u(t)\}$

226³

Solution



Laplace transform formula

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t) e^{-st} dt$$

$$\mathcal{L}\{t u(t)\} = \int_0^{\infty} t e^{-st} dt =$$

Note: $\int x e^{ax} dx = e^{ax} \left(\frac{x}{a} - \frac{1}{a^2} \right)$

$$\int_0^{\infty} t e^{-st} dt = e^{-st} \left(\frac{t}{-s} - \frac{1}{(-s)^2} \right) \Big|_0^{\infty} = e^{-st} \left(-\frac{t}{s} - \frac{1}{s^2} \right) \Big|_0^{\infty}$$

$$= \left[e^{-\infty} \left(-\frac{\infty}{s} - \frac{1}{s^2} \right) \right] - \left[e^{-0} \left(-\frac{0}{s} - \frac{1}{s^2} \right) \right]$$

$$= 0 - \left(-\frac{1}{s^2} \right) = \frac{1}{s^2}$$

$$1 \quad \delta(t)$$

$$\frac{1}{s} \quad u(t)$$

$$\frac{a}{s^2 + a^2} \quad \sin at$$

$$\frac{s}{s^2 + a^2} \quad \cos at$$

$$\frac{1}{s+a} \quad e^{-at}$$

Example problem: Calculate inverse Laplace Trans

$$a) f_1(s) = \frac{4}{s} \quad b) f_2(s) = 4 + \frac{5}{s} \quad c) f_3(s) = \frac{1}{s+6}$$

$$d) f_4(s) = \frac{8}{s+10} \quad e) f_5(s) = \frac{3}{s} + \frac{4}{s-2} - \frac{6}{s+3} \quad f) f_6(s) = \frac{1}{s^2+4}$$

Solution

$$a) f_1(s) = 4 \cdot \frac{1}{s} \Rightarrow f_1(t) = 4 u(t)$$

$$b) f_2(s) = 4 + \frac{5}{s} \Rightarrow f_2(t) = 4 \delta(t) + 5 u(t)$$

$$c) f_3(s) = \frac{1}{s+6} \Rightarrow f_3(t) = e^{-6t}$$

$$d) f_4(s) = \frac{8}{s+10} = 8 \frac{1}{s+10} \Rightarrow f_4(t) = 8 e^{-10t}$$

$$e) f_5(s) = \frac{3}{s} + 4 \frac{1}{s-2} - 6 \frac{1}{s+3} \Rightarrow f_5(t) = 3 u(t) + 4 e^{2t} - 6 e^{-3t}$$

$$f) f_6(s) = \frac{1}{s^2+4} = \frac{1}{2} \frac{2}{s^2+2^2} \Rightarrow \frac{1}{2} \sin 2t$$

Example problem: calculate $f(t)$

33 2

$$a) F_1(s) = \frac{2}{s} + \frac{3}{s+1}$$

$$b) F_2(s) = \frac{3}{s+2} + \frac{4}{s^2+9}$$

solution

$$a) F_1(s) = 2 \cdot \frac{1}{s} + 3 \frac{1}{s+1} \Rightarrow f_1(t) = 2 u(t) + 3 e^{-t}$$

$$b) F_2(s) = 3 \frac{1}{s+2} + 4 \frac{1}{3} \frac{3}{s^2+3^2}$$

$$\downarrow \qquad \qquad \qquad \downarrow$$
$$f_2(t) = 3 e^{-2t} + \frac{4}{3} \sin 3t$$

Example problem calculate $f(t)$

$$F_1(s) = \frac{10s}{s^2+9} + \frac{8}{s^2+9} = 10 \frac{s}{s^2+3^2} + \frac{8}{3} \frac{3}{s^2+3^2}$$

$$f_1(t) = 10 \cos 3t + \frac{8}{3} \sin 3t$$

Example problem calculate $f(t)$

$$a) F(s) = \frac{5s+4}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2} = \frac{A(s+2) + Bs}{s(s+2)} = \frac{(A+B)s + 2A}{s(s+2)}$$

$$\frac{5s+4}{s(s+2)} = \frac{(A+B)s + 2A}{s(s+2)}$$

$$\begin{pmatrix} 5 = A+B \\ 4 = 2A \end{pmatrix} \Rightarrow \begin{matrix} A = 2 \\ B = 3 \end{matrix}$$

$$F(s) = \frac{5s+4}{s(s+2)} = \frac{2}{s} + \frac{3}{s+2}$$

$$f(t) = 2 u(t) + 3 e^{-2t}$$

$$f(s) = \frac{9s^2 + 52s + 67}{s^3 + 9s^2 + 23s + 15}$$

$$s^3 + 9s^2 + 23s + 15 = 0 \Rightarrow \begin{aligned} s_1 &= -1 \\ s_2 &= -3 \\ s_3 &= -5 \end{aligned}$$

$$\frac{9s^2 + 52s + 67}{s^3 + 9s^2 + 23s + 15} = \frac{9s^2 + 52s + 67}{(s+1)(s+3)(s+5)} = \frac{A}{s+1} + \frac{B}{s+3} + \frac{C}{s+5}$$

Calculate A, B, C

$$A = 2$$

$$B = 3$$

$$C = 4$$

$$f(s) = \frac{9s^2 + 52s + 67}{s^3 + 9s^2 + 23s + 15} = \frac{2}{s+1} + \frac{3}{s+3} + \frac{4}{s+5}$$

$$f(t) = 2e^{-t} + 3e^{-3t} + 4e^{-5t}$$

Problem: How can we calculate A, B, C .

Residues: Partial Fraction Expansion.

$$f(z) = \frac{p(z)}{q(z)} = \frac{A}{z-a} + \frac{B}{z-b} + \frac{C}{z-c} + \dots$$

First residue Formula

$$A = \text{Res}(z=a) = \lim_{z \rightarrow a} (z-a) \frac{p(z)}{q(z)}$$

$$B = \text{Res}(z=b) = \lim_{z \rightarrow b} (z-b) \frac{p(z)}{q(z)}$$

$$C = \text{Res}(z=c) = \lim_{z \rightarrow c} (z-c) \frac{p(z)}{q(z)}$$

Second residue Formula

$$A = \text{Res}(z=a) = \lim_{z \rightarrow a} \frac{p(z)}{q'(z)}$$

$$B = \text{Res}(z=b) = \lim_{z \rightarrow b} \frac{p(z)}{q'(z)}$$

$$C = \text{Res}(z=c) = \lim_{z \rightarrow c} \frac{p(z)}{q'(z)}$$

Example AE-611

$$\frac{z+5}{z^3-5z^2-2z+24} = \frac{z+5}{(z-4)(z+2)(z-3)} = \frac{A}{z-4} + \frac{B}{z+2} + \frac{C}{z-3}$$

$$A = \lim_{z \rightarrow 4} (z-4) \frac{p(z)}{q(z)} = \lim_{z \rightarrow 4} (z-4) \frac{z+5}{(z-4)(z-3)(z+2)} = \lim_{z \rightarrow 4} \frac{z+5}{(z-3)(z+2)} = \frac{4+5}{(4-3)(4+2)} = \frac{9}{6} = 1.5$$

$$B = \lim_{z \rightarrow -2} (z-(-2)) \frac{p(z)}{q(z)} = \lim_{z \rightarrow -2} (z+2) \frac{z+5}{(z-4)(z-3)(z+2)} = \lim_{z \rightarrow -2} \frac{z+5}{(z-4)(z-3)} = \frac{-2+5}{(-2-4)(-2-3)} = \frac{3}{30} = 0.1$$

$$C = \lim_{z \rightarrow 3} (z-3) \frac{p(z)}{q(z)} = \lim_{z \rightarrow 3} \frac{z+5}{(z-4)(z+2)} = \frac{3+5}{(3-4)(3+2)} = -1.6$$

Using the Second residue Formula

In our problem $p(z)=z+5$ $q(z)=z^3-5z^2-2z+24$ and $q'(z)=3z^2-10z-2$

$$A = \lim_{z \rightarrow 4} \frac{z+5}{3z^2-10z-2} = \frac{4+5}{3 \cdot 4^2 - 10 \cdot 4 - 2} = \frac{9}{6} = 1.5$$

$$B = \lim_{z \rightarrow -2} \frac{z+5}{3z^2-10z-2} = \frac{-2+5}{3 \cdot (-2)^2 - 10 \cdot (-2) - 2} = \frac{3}{30} = 0.1$$

$$C = \lim_{z \rightarrow 3} \frac{z+5}{3z^2-10z-2} = \frac{3+5}{3 \cdot (3)^2 - 10 \cdot (3) - 2} = \frac{8}{-5} = -1.6$$

Note we get the same result by classical method

$$\begin{aligned} \frac{z+5}{z^3-5z^2-2z+24} &= \frac{A}{z-4} + \frac{B}{z+2} + \frac{C}{z-3} \\ &= \frac{A(z+2)(z-3) + B(z-4)(z-3) + C(z-4)(z+2)}{(z-4)(z+2)(z-3)} \\ &= \frac{A(z^2-z-6) + B(z^2-7z-12) + C(z^2-2z-8)}{(z-4)(z+2)(z-3)} \\ &= \frac{z^2(A+B+C) + z(-7A-B-2C) + 12A-6B-8C}{(z-4)(z+2)(z-3)} \end{aligned}$$

$$\begin{aligned} z^2(A+B+C) + z(-7A-B-2C) + 12A-6B-8C &= z+5 \\ A+B+C &= 0, \quad -7A-B-2C=1, \quad 12A-6B-8C=5 \end{aligned}$$

Solving for A,B,C we get $A=1.5$ $B=0.1$ $C=-1.6$

$$\frac{z+5}{z^3-5z^2-2z+24} = \frac{1.5}{z-4} + \frac{0.1}{z+2} + \frac{-1.6}{z-3}$$

Example AE-612

$$\frac{2z+12}{z^2+2z+2} = \frac{2z+12}{[z-(-1+i)][z-(-1-i)]} = \frac{2z+12}{(z+1-i)(z+1+i)} = \frac{A}{z+1-i} + \frac{B}{z+1+i}$$

Using the first residue formula

$$A = \lim_{z \rightarrow -1+i} (z+1-i) \frac{2z+12}{(z+1-i)(z+1+i)} = \lim_{z \rightarrow -1+i} \frac{2z+12}{z+1+i} = \frac{2(-1+i)+12}{((-1+i)+1+i)} = \frac{2i+10}{2i} = \frac{2i}{2i} + \frac{10}{2i} = 1-5i$$

$$B = \lim_{z \rightarrow -1-i} (z+1+i) \frac{2z+12}{(z+1-i)(z+1+i)} = \lim_{z \rightarrow -1-i} \frac{2z+12}{z+1-i} = 1+5i$$

Using the second residue formula

$$p(z)=2z+12, \quad q(z)=z^2+2z+2, \quad q'(z)=2z+2$$

$$A = \lim_{z \rightarrow -1+i} \frac{p(z)}{q'(z)} = \frac{2z+12}{2z+2} = \frac{2(-1+i)+12}{2(-1+i)+2} = 1-5i$$

$$B = \lim_{z \rightarrow -1-i} \frac{p(z)}{q'(z)} = \frac{2z+12}{2z+2} = \frac{2(-1-i)+12}{2(-1-i)+2} = 1+5i$$

Using classical Method

$$\frac{2z+12}{z^2+2z+2} = \frac{A}{z+1-i} + \frac{B}{z+1+i} = \frac{A(z+1+i) + B(z+1-i)}{(z+1-i)(z+1+i)}$$

$$(A+B)=2 \quad A(1+i)+B(1-i)=12 \quad \text{solution is } A=1-5i, \quad B=1+5i$$

$$\text{Result } \frac{2z+12}{z^2+2z+2} = \frac{1-5i}{z+1-i} + \frac{1+5i}{z+1+i}$$

EX.- 28

$$f(s) = \frac{s+3}{(s+1)(s+2)} = \frac{s+3}{s^2+3s+2} \quad f(t) = ? \quad 335$$

Solution

$$\frac{s+3}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$A = (s+1) f(s) \Big|_{s=-1} = (s+1) \frac{s+3}{(s+1)(s+2)} \Big|_{s=-1} = \frac{s+3}{s+2} \Big|_{s=-1} = \frac{-1+3}{-1+2} = 2$$

$$\boxed{A=2}$$

$$B = (s+2) f(s) \Big|_{s=-2} = (s+2) \frac{s+3}{(s+1)(s+2)} \Big|_{s=-2} = \frac{s+3}{s+1} \Big|_{s=-2} = \frac{-2+3}{-2+1} = -1$$

$$B = -1$$

$$f(s) = \frac{s+3}{(s+1)(s+2)} = \frac{2}{s+1} + \frac{-1}{s+2}$$

$$f(t) = 2e^{-t} - e^{-2t}$$

complex roots:

S-341

$$\frac{As+B}{s^2+ms+n} = \frac{c_1}{s-s_1} + \frac{c_2}{s-s_2}$$

c_1, c_2, s_1, s_2 are complex
and $c_2 = c_1^*$ $s_2 = s_1^*$

Examples

$$\frac{10s+14}{s^2+6s+14} = \frac{5+2i}{s-(-3+4i)} + \frac{5-2i}{s-(-3-4i)}$$

$$\frac{6s+2}{s^2+6s+14} = \frac{3+2i}{s-(-3+4i)} + \frac{3-2i}{s-(-3-4i)}$$

$$\frac{4s-22}{s^2+10s+34} = \frac{2+7i}{s-(-5+3i)} + \frac{2-7i}{s-(-5-3i)}$$

$$\frac{4s+62}{s^2+10s+34} = \frac{2-7i}{s-(-5+3i)} + \frac{2+7i}{s-(-5-3i)}$$

$$\frac{As+B}{s^2+ms+n} = \frac{a+bi}{s-(x+iy)} + \frac{a-bi}{s-(x-iy)}$$

$$\mathcal{L}^{-1} \left\{ \frac{a+bi}{s-(x+iy)} + \frac{a-bi}{s-(x-iy)} \right\} = (a+bi) e^{(x+iy)t} + (a-bi) e^{(x-iy)t}$$

$$= (a+bi) e^{xt} e^{iyt} + (a-bi) e^{xt} e^{-iyt} = e^{xt} \left((a+bi) e^{iyt} + (a-bi) e^{-iyt} \right)$$

$$= e^{xt} \left[(a+bi) (\cos yt + i \sin yt) + (a-bi) (\cos yt - i \sin yt) \right]$$

$$= e^{xt} \left[a \cos yt + ai \sin yt + bi \cos yt - b \sin yt + a \cos yt - a i \sin yt - bi \cos yt - b \sin yt \right]$$

$$\mathcal{L}^{-1} \left\{ \frac{a+bi}{s-(x+iy)} + \frac{a-bi}{s-(x-iy)} \right\} = e^{xt} [2a \cos yt - 2b \sin yt]$$

$$\mathcal{L}^{-1} \left\{ \frac{10s+14}{s^2+6s+25} \right\} = \mathcal{L}^{-1} \left\{ \frac{5+2i}{s-(-3+4i)} + \frac{5-2i}{s-(-3-4i)} \right\} = e^{-3t} (2 \times 5 \cos 4t - 2 \times 2 \sin 4t)$$

$$= e^{-3t} (10 \cos 4t - 4 \sin 4t)$$

$$\mathcal{L}^{-1} \left\{ \frac{6s+2}{s^2+6s+25} \right\} = \mathcal{L}^{-1} \left\{ \frac{3+2i}{s-(-3+4i)} + \frac{3-2i}{s-(-3-4i)} \right\} = e^{-3t} (2 \times 3 \cos 4t - 2 \times 2 \sin 4t)$$

$$= e^{-3t} (6 \cos 4t - 4 \sin 4t)$$

$$\mathcal{L}^{-1} \left\{ \frac{4s-22}{s^2+10s+34} \right\} = \mathcal{L}^{-1} \left\{ \frac{2+7i}{s-(-5+3i)} + \frac{2-7i}{s-(-5-3i)} \right\} = e^{-5t} (2 \times 2 \cos 3t - 2 \times 7 \sin 3t)$$

$$= e^{-5t} (4 \cos 3t - 14 \sin 3t)$$

(+3i) (+7i)

$$\mathcal{L}^{-1} \left\{ \frac{4s+62}{s^2+10s+34} \right\} = \mathcal{L}^{-1} \left\{ \frac{2-7i}{s-(-5+3i)} + \frac{2+7i}{s-(-5-3i)} \right\} = e^{-5t} (2 \times 2 \cos 3t - 2 \times (-7) \sin 3t)$$

$$= e^{-5t} (4 \cos 3t + 14 \sin 3t)$$

(+3i) (-7i)

$$\mathcal{L}^{-1} \left\{ \frac{4s+22}{s^2-10s+34} \right\} = \mathcal{L}^{-1} \left\{ \frac{2-7i}{s-(5+3i)} + \frac{2+7i}{s-(5-3i)} \right\} = e^{5t} (2 \times 2 \cos 4t - 2 \times (-7) \sin 4t)$$

$$= e^{5t} (4 \cos 4t + 14 \sin 4t)$$

$$\mathcal{L}\{e^{at} \cos bt\} = \frac{s-a}{(s-a)^2 + b^2}$$

$$\mathcal{L}\{e^{-at} \cos bt\} = \frac{s+a}{(s+a)^2 + b^2} \quad 343$$

$$\mathcal{L}\{e^{at} \sin bt\} = \frac{b}{(s-a)^2 + b^2}$$

$$\mathcal{L}\{e^{-at} \sin bt\} = \frac{b}{(s+a)^2 + b^2}$$

EX-34

$$\frac{10s+14}{s^2+6s+25}$$

$$s^2+6s+25=0 \rightarrow \begin{matrix} s_1 = -3+4i \\ s_2 = -3-4i \end{matrix}$$

$$s^2+6s+25 = (s+3)^2 + 4^2$$

$$\begin{aligned} 14 &= 10 \times 3 + x \\ x &= 14 - 30 = -16 \end{aligned}$$

$$\frac{10s+14}{s^2+6s+25} = \frac{10s+14}{(s+3)^2+4^2} = 10 \frac{(s+3)}{(s+3)^2+4^2} + \frac{-16}{(s+3)^2+4^2}$$

$$= 10 \frac{s+3}{(s+3)^2+4^2} + (-4) \frac{4}{(s+3)^2+4^2}$$



$$10 e^{-3t} \cos 4t + (-4) e^{-3t} \sin 4t$$

$$= e^{-3t} (10 \cos 4t - 4 \sin 4t)$$

EX-35

$$\begin{aligned} 2 &= 6 \times 3 + x \\ 2 - 18 &= x \end{aligned}$$

$$\frac{6s+2}{s^2+6s+25} = \frac{6s+2}{(s+3)^2+4^2} = 6 \frac{s+3}{(s+3)^2+4^2} + \frac{-16}{(s+3)^2+4^2} = 6 \frac{s+3}{(s+3)^2+4^2} - 4 \frac{4}{(s+3)^2+4^2}$$



$$6 e^{-3t} \cos 4t - 4 e^{-3t} \sin 4t$$

$$= e^{-3t} (6 \cos 4t - 4 \sin 4t)$$

EX-36

8-344

$$\mathcal{L}^{-1} \left\{ \frac{4s-22}{s^2+10s+34} \right\} = ?$$

$$s^2 + 10s + 34 = 0 \quad \begin{array}{l} \rightarrow s_1 = -5 + 3i \\ \rightarrow s_2 = -5 - 3i \end{array}$$

$$\frac{4s-22}{s^2+10s+34} = \frac{4s-22}{(s+5)^2+3^2} = 4 \frac{s+5}{(s+5)^2+3^2} + \frac{-42}{(s+5)^2+3^2}$$

$-22 = 4 \times 5 + x$
 $-22 - 20 = x$

$$= 4 \frac{s+5}{(s+5)^2+3^2} - 14 \frac{3}{(s+5)^2+3^2}$$

$$\downarrow$$

$$-4e^{-5t} \cos 3t - 14e^{-5t} \sin 3t$$

$$= e^{-5t} (4 \cos 3t - 14 \sin 3t)$$

EX-37

$$\mathcal{L}^{-1} \left\{ \frac{4s+22}{s^2-10s+34} \right\}$$

$$s^2 - 10s + 34 = 0 \quad \begin{array}{l} \rightarrow s_1 = 5 + 3i \\ \rightarrow s_2 = 5 - 3i \end{array}$$

$$\frac{4s+22}{s^2-10s+34} = \frac{4s+22}{(s-5)^2+3^2} = 4 \frac{s-5}{(s-5)^2+3^2} + \frac{42}{(s-5)^2+3^2}$$

$22 = 4 \times (-5) + x$
 $22 + 20 = x$

$$= 4 \frac{s-5}{(s-5)^2+3^2} + 14 \frac{3}{(s-5)^2+3^2}$$

$$\downarrow$$

$$4e^{5t} \cos 3t + 14e^{5t} \sin 3t$$

$$= e^{5t} (4 \cos 3t + 14 \sin 3t)$$

EX-41

s-345

$$F(s) = \frac{13s^2 + 82s + 122}{s^3 + 3s^2 + 26s + 24} = \frac{1}{s+4} + \frac{7}{s+3} + \frac{5}{s+2}$$

$$f(t) = e^{-4t} + 7e^{-3t} + 5e^{-2t}$$

EX-42

$$F(s) = \frac{17s^4 - 16s^3 + 497s^2 - 1200s + 10978}{s^5 - 4s^4 + 23s^3 - 416s^2 + 2966s - 4420}$$

$$= \frac{3}{s - (-4+7i)} + \frac{3}{s - (-4-7i)} + \frac{2+7i}{s - (5+3i)} + \frac{2+7i}{s - (5-3i)} + \frac{7}{s-2}$$

$$f(t) = e^{-4t} (2 \times 3 \cos 7t - 2 \times 0 \sin 7t) + e^{5t} (2 \times 2 \cos 3t - 2 \times (-7) \sin 3t) + 7e^{2t}$$

$$= 6e^{-4t} \cos 7t + e^{5t} (4 \cos 3t + 14 \sin 3t) + 7e^{2t}$$

Repeated real roots

347

$$F(s) = \frac{10}{(s+2)} \rightarrow f(t) = 10 e^{-2t}$$

$$f(s) = \frac{10}{(s+2)^2} \rightarrow f(t) = 10 t e^{-2t}$$

$$F(s) = \frac{10}{(s+2)^3} \rightarrow f(t) = 10 t^2 e^{-2t}$$

$$F(s) = \frac{2}{s+4} + \frac{3}{(s+4)^2} \rightarrow f(t) = 2 e^{-4t} + 3 t e^{-4t}$$

$$f(s) = \frac{2}{s+2} + \frac{3}{(s+4)^2} + \frac{5}{(s+1)^3} \rightarrow f(t) = 2 e^{-2t} + 3 t e^{-4t} + 5 t^2 e^{-t}$$

$$F(s) = \frac{1}{s} + \frac{2}{s^2} + \frac{3}{s+1} \rightarrow f(t) = u(t) + t u(t) + 3 e^{-t}$$

$$F(s) = \frac{100(s+25)}{s(s+5)^3} = \frac{A}{s} + \frac{B}{(s+5)} + \frac{C}{(s+5)^2} + \frac{D}{(s+5)^3}$$

$$A = 20 \quad B = -400 \quad C = -100 \quad D = -20$$

$$F(s) = \frac{20}{s} - \frac{400}{s+5} - \frac{100}{(s+5)^2} - \frac{20}{(s+5)^3}$$

$$f(t) = 20 u(t) - 400 e^{-5t} - 100 t e^{-5t} - 20 t^2 e^{-5t}$$

How can we calculate A, B, C, D

$$F(s) = \frac{ms+n}{(s+a)^2(s+b)} = \frac{A}{s+a} + \frac{B}{(s+a)^2} + \frac{C}{s+b}$$

348

$$C = (s+b) F(s) \Big|_{s=-b} = (s+b) \frac{ms+n}{(s+a)^2(s+b)} \Big|_{s=-b} = \frac{ms+n}{(s+a)^2} \Big|_{s=-b}$$

$$= \frac{m(-b)+n}{(-b+a)^2}$$

$$B = (s+a)^2 F(s) \Big|_{s=-a} = (s+a)^2 \frac{ms+n}{(s+a)^2(s+b)} \Big|_{s=-a} = \frac{ms+n}{s+b} \Big|_{s=-a} = \frac{-ma+n}{-a+b}$$

$$A = \frac{d}{ds} \left[(s+a)^2 F(s) \right] \Big|_{s=-a} = \frac{d}{ds} \left[\frac{ms+n}{s+b} \right] = \frac{(ms+n)'(s+b) - (ms+n)(s+b)'}{(s+b)^2}$$

$$= \frac{m(s+b) - (ms+n)}{(s+b)^2}$$

$$f(s) = \frac{6s^2+17s+14}{(s+1)^2(s+2)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+2}$$

$$C = \frac{6s^2+17s+14}{(s+1)^2} \Big|_{s=-2} = \frac{6(-2)^2+17(-2)+14}{(-2+1)^2} = 4$$

$$B = \frac{6s^2+17s+14}{s+2} \Big|_{s=-1} = \frac{6(-1)^2+17(-2)+14}{-1+2} = 3$$

$$A = \frac{d}{ds} \left[\frac{6s^2+17s+14}{s+2} \right] \Big|_{s=-1} = \frac{(12s+17)(s+2) - 1(6s^2+17s+14)}{(s+2)^2} \Big|_{s=-1} = 2$$

$$\frac{15z^3+1}{(z+2)^3(z-1)^2(z+10)} = \frac{A_1}{(z+2)} + \frac{A_2}{(z+2)^2} + \frac{A_3}{(z+2)^3} + \frac{B_1}{(z-1)} + \frac{B_2}{(z-1)^2} + \frac{C}{(z+10)}$$

$$C = (z+10)F(z)\Big|_{z=-10} = \frac{15z^3+1}{(z+2)^3(z-1)^2}\Big|_{z=-10} = \frac{15(-10)^3+1}{(-10+2)^3(-10-1)^2} = 0.2421$$

$$B_2 = (z-1)^2 F(z)\Big|_{z=1} = \frac{15z^3+1}{(z+2)^3(z+10)}\Big|_{z=1} = \frac{15(1)^3+1}{(1+2)^3(1+10)} = 0.0539$$

$$B_1 = \frac{d}{dz}[(z-1)^2 F(z)]\Big|_{z=1} = \frac{d}{dz}\left[\frac{15z^3+1}{(z+2)^3(z+10)}\right]\Big|_{z=1}$$

$$p=15z^3+1 \quad p'=45z^2$$

$$q=(z+2)^3(z+10) \quad q'=3(z+2)^2(z+10) + (z+2)^3 \cdot 1$$

$$q'=4z^3+48z^2+144z+128$$

$$\left(\frac{p}{q}\right)' = \left(\frac{p'q - pq'}{q^2}\right)$$

$$= \frac{-15z^6 + 1080z^4 + 3836z^3 + 3552z^2 - 144z - 128}{(z+2)^6(z+10)^2}$$

$$\frac{-15z^6 + 1080z^4 + 3836z^3 + 3552z^2 - 144z - 128}{(z+2)^6(z+10)^2}\Big|_{z=1}$$

$$\frac{-15(1)^6 + 1080(1)^4 + 3836(1)^3 + 3552(1)^2 - 144(1) - 128}{(1+2)^6(1+10)^2} = 0.092$$

$$B_1 = 0.092$$

$$A_3 = (z+2)^3 F(z)\Big|_{z=-2} = \frac{15z^3+1}{(z-1)^2(z+10)}\Big|_{z=-2} = \frac{15(-2)^3+1}{(-2-1)^2(-2+10)} = -1.6528$$

$$A_2 = \frac{d}{dz}[(z+2)^3 F(z)]\Big|_{z=-2} = \frac{d}{dz}\left[\frac{15z^3+1}{(z-1)^2(z+10)}\right]\Big|_{z=-2}$$

$$p=15z^3+1 \quad p'=45z^2$$

$$q=(z-1)^2(z+10) \quad q'=2(z-1)(z+10) + (z-1)^2 \cdot 1$$

$$q'=3z^2+16z-19$$

$$A_2 = 1.604$$

$$A_1 = \frac{d^2}{dz^2}\left[\frac{15z^3+1}{(z-1)^2(z+10)}\right]\Big|_{z=-2} = -0.33$$

$$\frac{15z^3+1}{(z+2)^3(z-1)^2(z+10)} =$$

$$\frac{-0.33}{z+2} + \frac{1.604}{(z+2)^2} + \frac{-1.652}{(z+2)^3} + \frac{0.092}{z-1}$$

$$+ \frac{0.0539}{(z-1)^2} + \frac{0.2421}{z+10}$$

Improper Rational functions

$$f(s) = \frac{s^2 + 7s + 5}{s + 3} \rightarrow \text{improper} \quad \begin{array}{l} \text{numerator} \rightarrow s^2 \quad (2) \\ \text{denominator} \rightarrow s \quad (1) \end{array}$$

$2 > 1$

$$\begin{array}{r} s+4 \\ s+3 \overline{) s^2+7s+5} \\ \underline{s^2+3s} \\ 0 4s+5 \\ \underline{4s+12} \\ -7 \end{array}$$

$$\frac{s^2 + 7s + 5}{s + 3} = s + 4 + \frac{-7}{s + 3}$$

$$\begin{array}{r} 3 \\ 5 \overline{) 17} \\ \underline{15} \\ 2 \end{array}$$

$$\frac{17}{5} = 3 + \frac{2}{5}$$

$$f(t) = \delta(t) \rightarrow f(s) = 1$$

$$f(t) = \frac{d}{dt} \delta(t) \rightarrow f(s) = s$$

$$f(t) = \frac{d^2}{dt^2} \delta(t) \rightarrow f(s) = s^2$$

$$f(s) = \frac{s^2 + 7s + 5}{s + 3} = s + 4 + \frac{-7}{s + 3}$$

$$f(t) = \frac{d}{dt} \delta(t) + 4\delta(t) - 7e^{-3t}$$

$$F(s) = \frac{s^3 + 7s^2 + 9s + 7}{s^2 + 3s + 2}$$

$$\begin{array}{r}
 s + 4 \\
 \hline
 s^2 + 3s + 2 \left| \begin{array}{l} s^3 + 7s^2 + 9s + 7 \\ s^3 + 3s^2 + 2s \\ \hline 4s^2 + 7s + 7 \\ 4s^2 + 12s + 8 \\ \hline -5s - 1 \end{array} \right.
 \end{array}$$

$$F(s) = \frac{s^3 + 7s^2 + 9s + 7}{s^2 + 3s + 2} = s + 4 + \frac{-5s - 1}{s^2 + 3s + 2}$$

$$\frac{-5s - 1}{s^2 + 3s + 2} = \frac{-5s - 1}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1}$$

$$A = \left. \frac{-5s - 1}{s+1} \right|_{s=-2} = \frac{-5(-2) - 1}{-2+1} = -9$$

$$B = \left. \frac{-5s - 1}{s+2} \right|_{s=-1} = \frac{-5(-1) - 1}{-1+2} = 4$$

$$F(s) = s + 4 + \frac{-9}{s+2} + \frac{4}{s+1}$$

$$f(t) = \frac{d}{dt} \delta(t) + 4\delta(t) - 9e^{-2t} + 4e^{-t}$$

$$F(s) = \frac{s^4 + 13s^3 + 66s^2 + 200s + 300}{s^2 + 9s + 20}$$

952

$$\begin{array}{r}
 s^2 + 4s + 10 \\
 s^2 + 9s + 20 \overline{) s^4 + 13s^3 + 66s^2 + 200s + 300} \\
 \underline{s^4 + 9s^3 + 20s^2} \\
 4s^3 + 46s^2 + 200s + 300 \\
 \underline{4s^3 + 36s^2 + 80s} \\
 10s^2 + 120s + 300 \\
 \underline{10s^2 + 90s + 200} \\
 30s + 100
 \end{array}$$

$$F(s) = s^2 + 4s + 10 + \frac{30s + 100}{s^2 + 9s + 20}$$

$$\frac{30s + 100}{s^2 + 9s + 20} = \frac{30s + 100}{(s+4)(s+5)} = \frac{-20}{s+4} + \frac{50}{s+5}$$

$$F(s) = s^2 + 4s + 10 + \frac{-20}{s+4} + \frac{50}{s+5}$$

$$f(t) = \frac{d^2}{dt^2} \delta(t) + 4 \frac{d}{dt} \delta(t) + 10 \delta(t) - 20 e^{-4t} + 50 e^{-5t}$$

An Inductor in the s Domain

Figure 13.2 shows an inductor carrying an initial current of I_0 amperes. The time-domain equation that relates the terminal voltage to the terminal current is

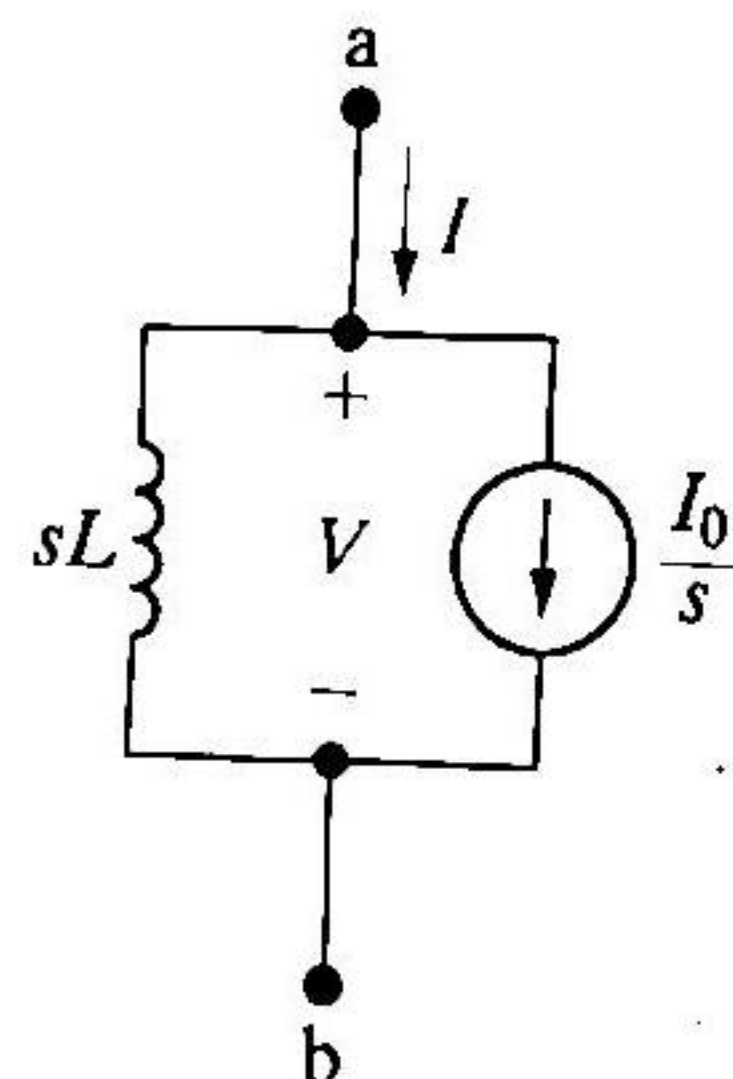
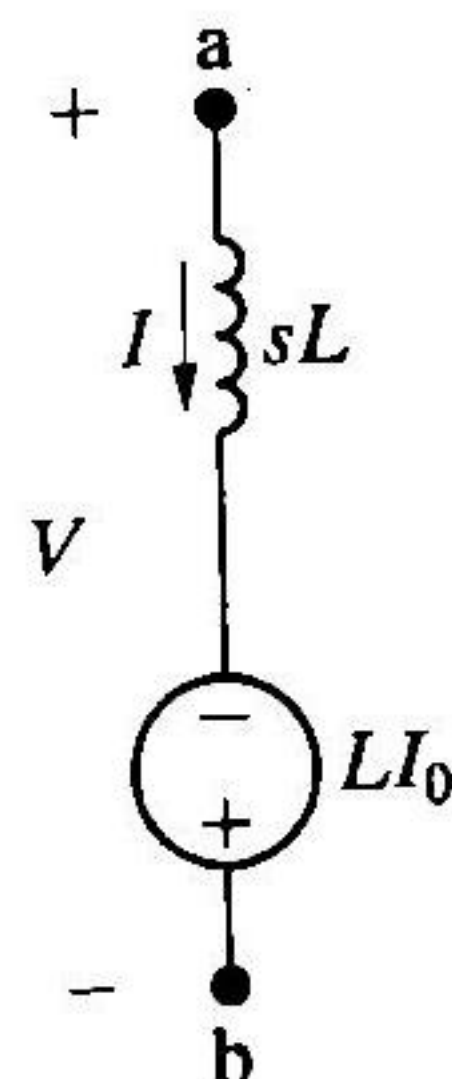
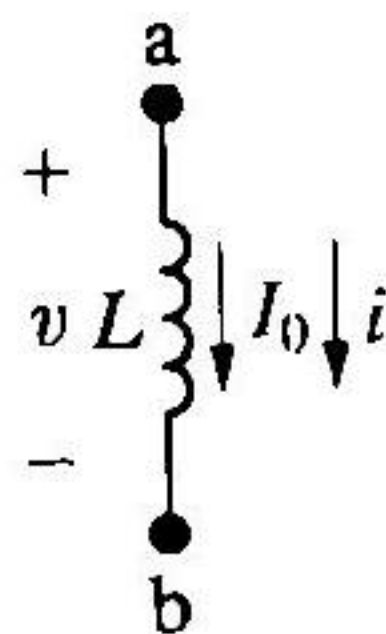
$$v = L \frac{di}{dt} \quad (13.3)$$

The Laplace transform of Eq. 13.3 gives

$$V = L[sI - i(0^-)] = sLI - LI_0. \quad (13.4)$$

$$V + LI_0 = sLI$$

$$I = \frac{V + LI_0}{sL} = \frac{V}{sL} + \frac{I_0}{s} \quad (13.5)$$



A Capacitor in the s Domain

An initially charged capacitor also has two s -domain equivalent circuits. Figure 13.6 shows a capacitor initially charged to V_0 volts. The terminal current is

$$i = C \frac{dv}{dt} \quad (13.6)$$

Transforming Eq. 13.6 yields

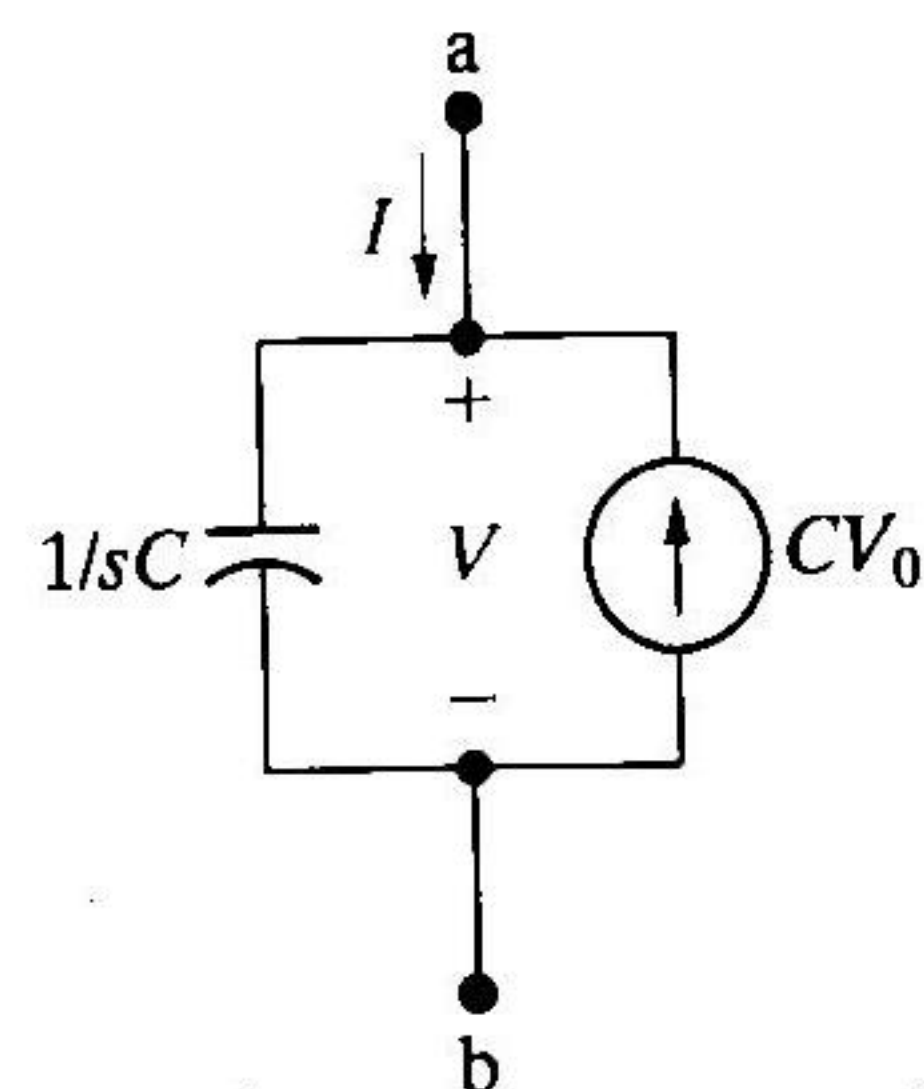
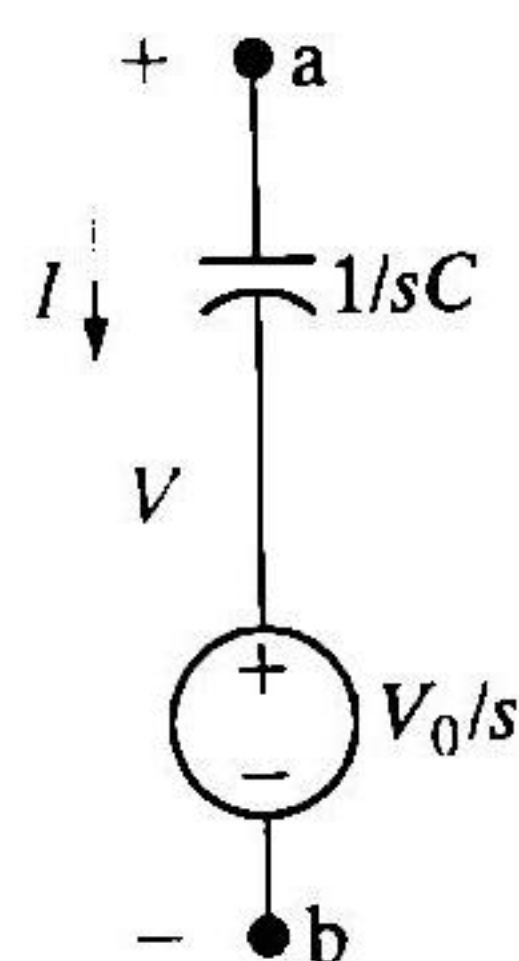
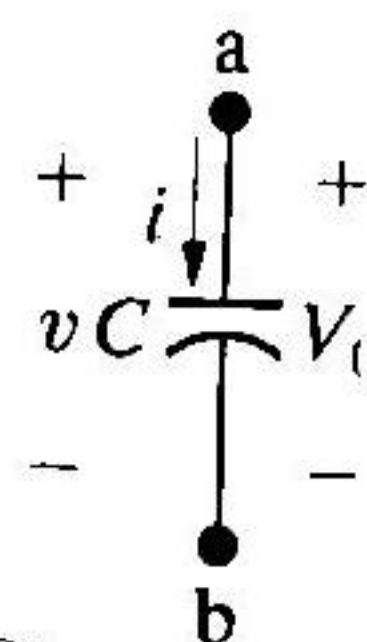
$$I = C[sV - v(0^-)]$$

or

$$I = sCV - CV_0, \quad (13.7)$$

$$I + CV_0 = sCV$$

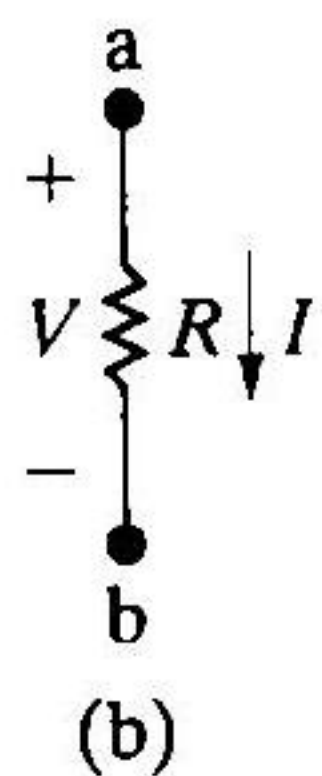
$$V = \left(\frac{1}{sC} \right) I + \frac{V_0}{s} \quad (13.8)$$

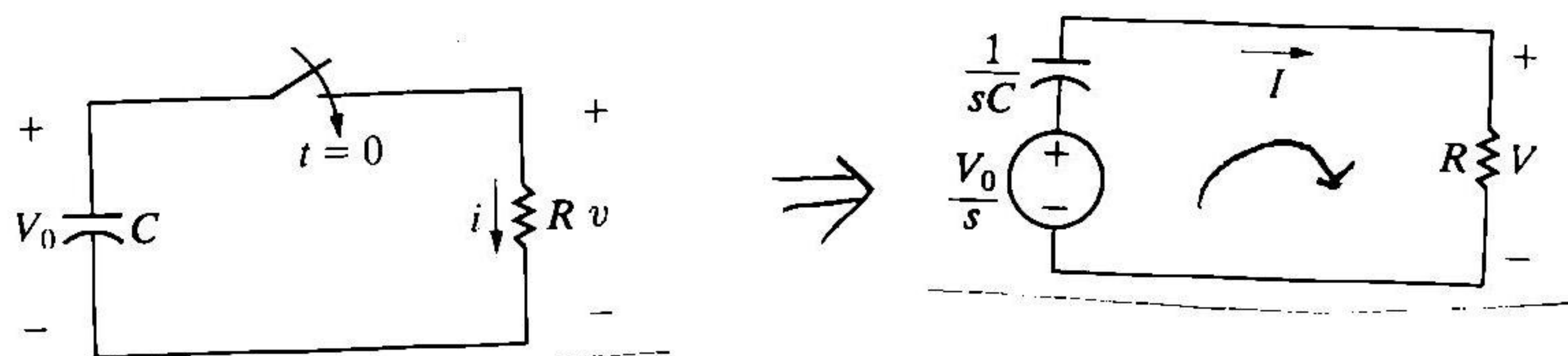


A Resistor in the s Domain

$$v = Ri$$

$$V = RI$$





$$-\frac{V_0}{s} + \left(\frac{1}{sC}\right) I + RI = 0$$

$$\left[\frac{1}{sC} + R\right] I = \frac{V_0}{s}$$

$$\frac{1 + R s C}{s C} I = \frac{V_0}{s}$$

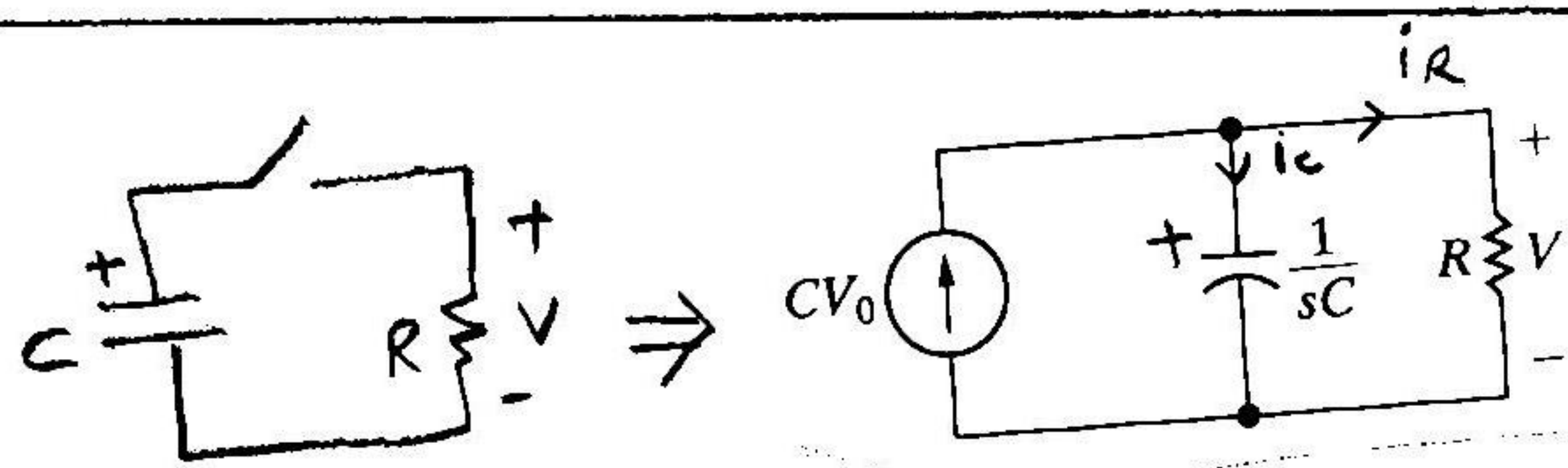
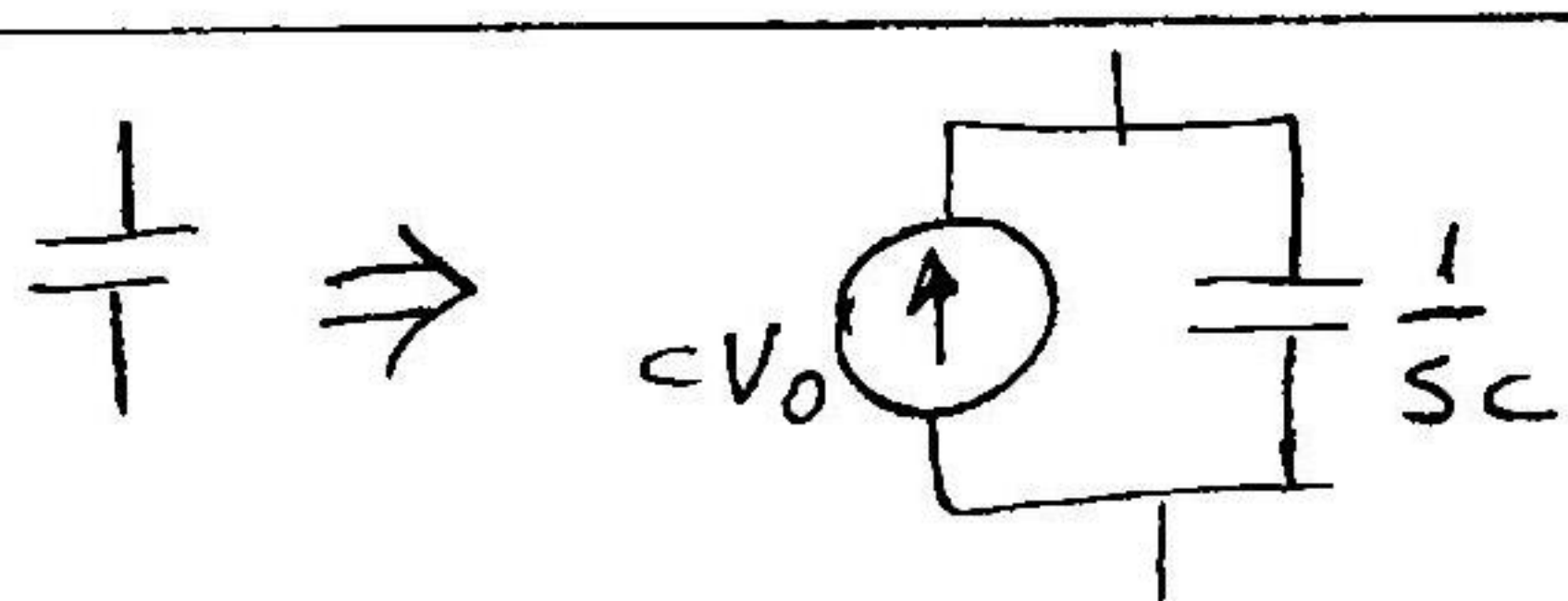
$$I = \frac{V_0}{s} \frac{sC}{1 + R s C} = \frac{V_0 C}{1 + R s C} = \frac{V_0/R}{s + \frac{1}{RC}}$$

$$F(s) = \frac{a}{s+b} \longrightarrow f(t) = a e^{-bt}$$

$$I(s) = \frac{V_0/R}{s + \frac{1}{RC}} \longrightarrow i(t) = \frac{V_0}{R} e^{-\frac{1}{RC}t}$$

The voltage across the resistor is $V(t) = Ri(t)$

$$V(t) = R i(t) = V_0 e^{-\frac{1}{RC}t}$$



node equation $cV_0 = i_C + i_R$

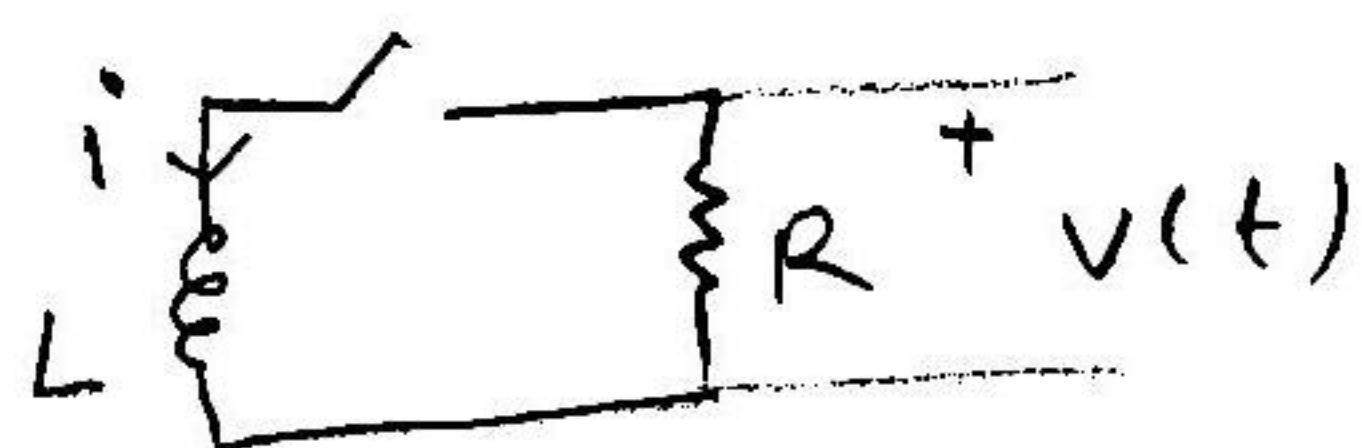
$$cV_0 = \frac{V}{\frac{1}{sC}} + \frac{V}{R} \Rightarrow cV_0 = sCV + \frac{V}{R}$$

$$cV_0 = \left(sC + \frac{1}{R}\right) V \Rightarrow V = \frac{cV_0}{sC + \frac{1}{R}} = \frac{cV_0/c}{s + \frac{1}{RC}} = \frac{V_0}{s + \frac{1}{RC}}$$

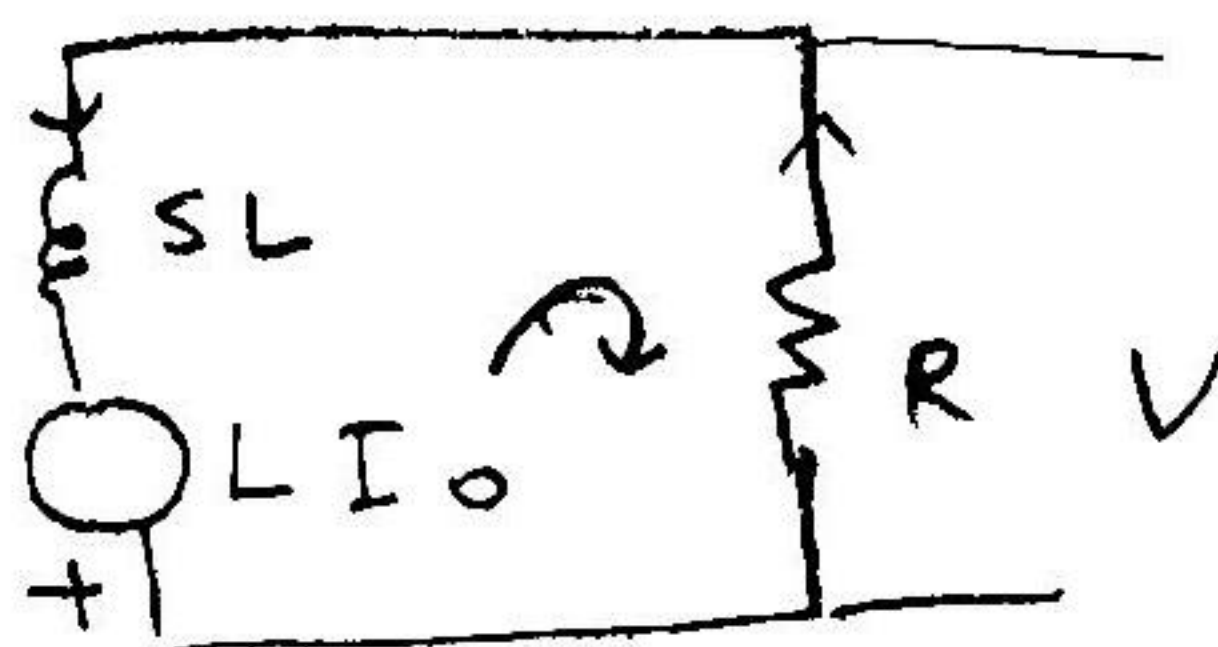
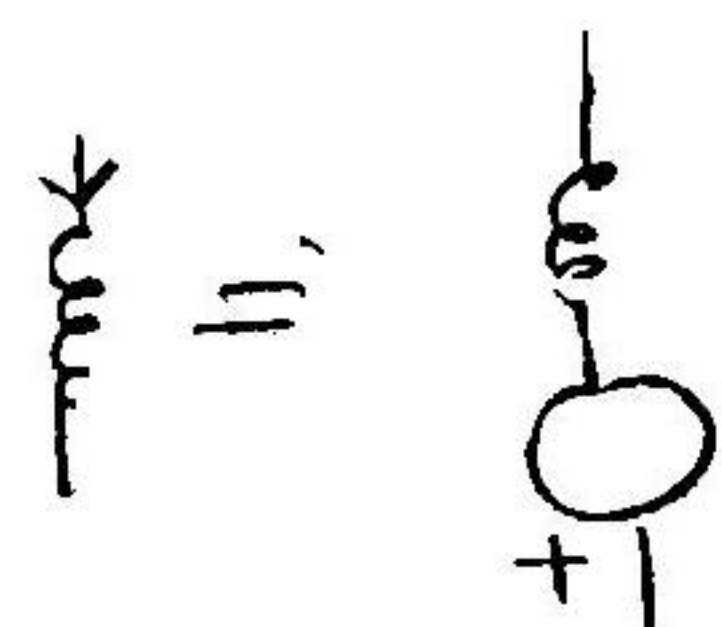
$$V(s) = \frac{V_0}{s + \frac{1}{RC}} \Rightarrow \boxed{v(t) = V_0 e^{-\frac{1}{RC}t}}$$

Example Problem: Find $i(t)$, $v(t)$

373



Solution



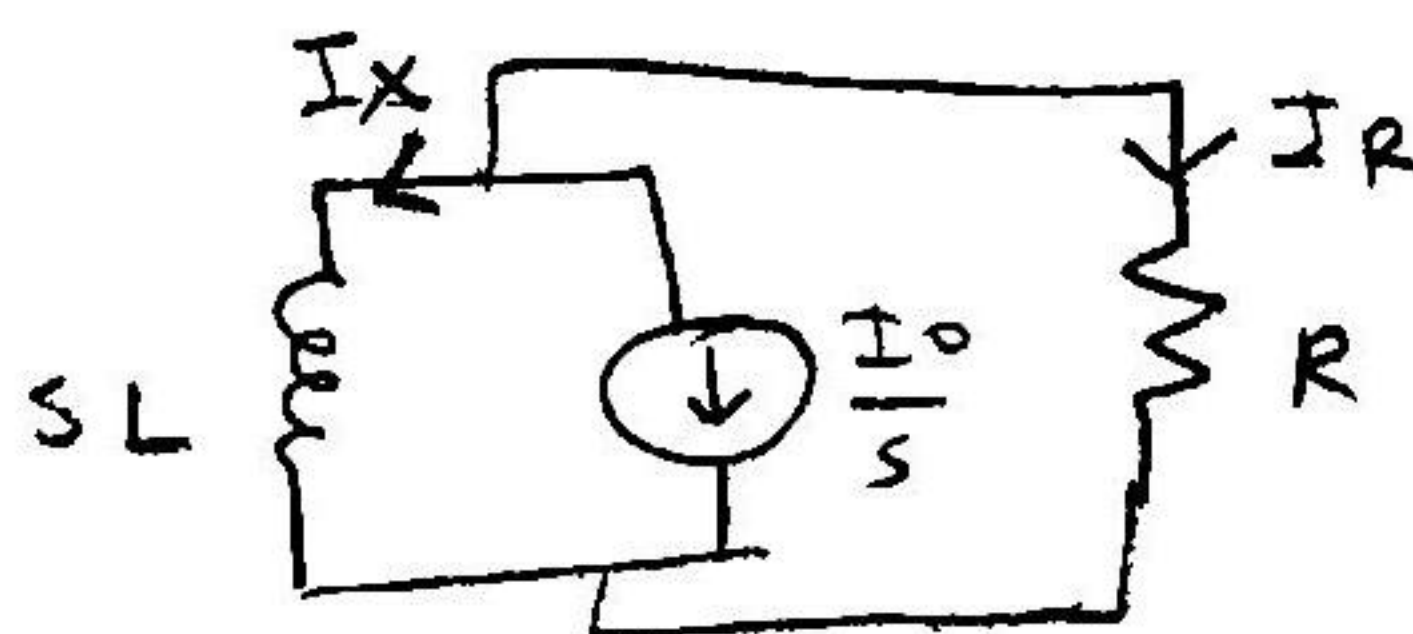
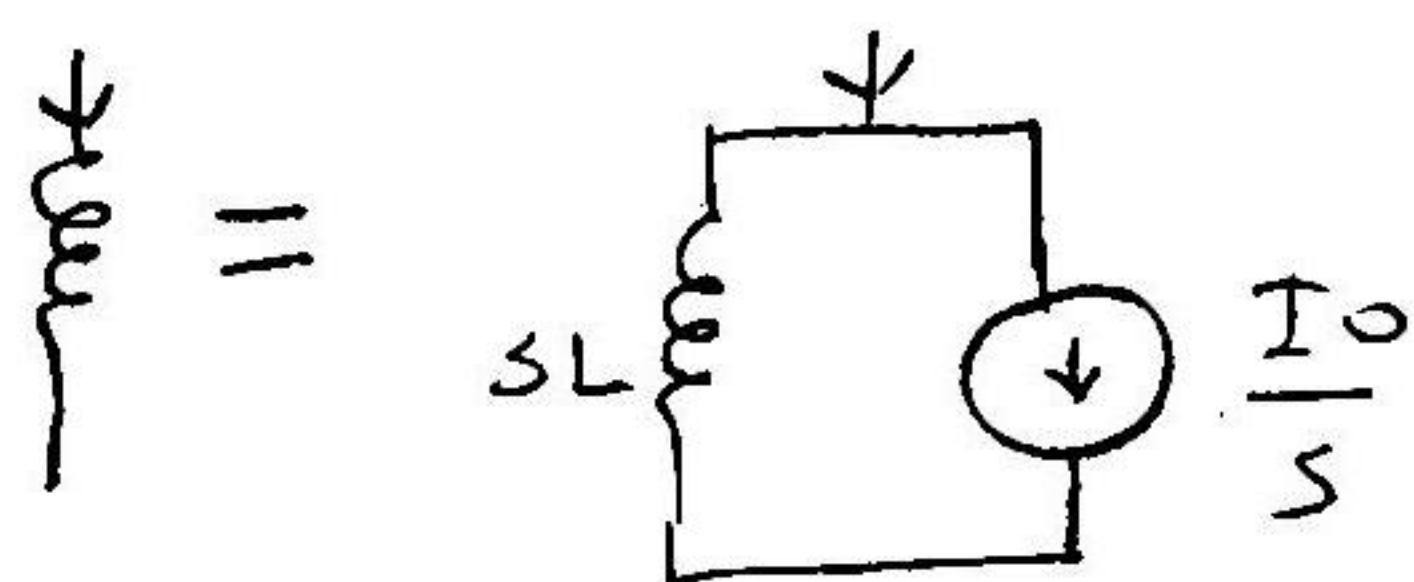
$$+LI_0 - sLI - RI = 0$$

$$I(sL + R) = LI_0 \Rightarrow I = \frac{LI_0}{sL + R} = \frac{I_0}{s + \frac{R}{L}}$$

$$I(s) = \frac{I_0}{s + \frac{R}{L}} \Rightarrow i(t) = I_0 e^{-\frac{R}{L}t}$$

$$v(t) = -Ri(t) = -RI_0 e^{-\frac{R}{L}t}$$

Second method



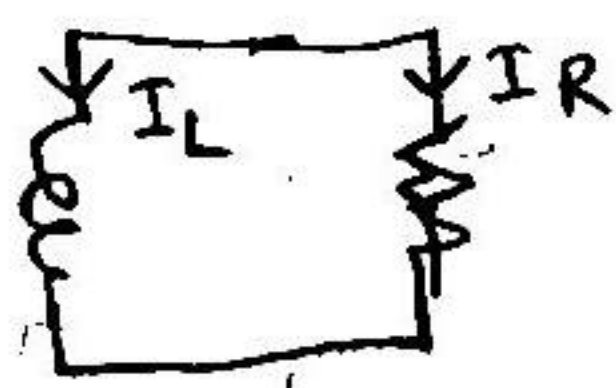
node equation

$$I_X + \frac{I_0}{s} + I_R = 0$$

$$\frac{V}{sL} + \frac{I_0}{s} + \frac{V}{R} = 0$$

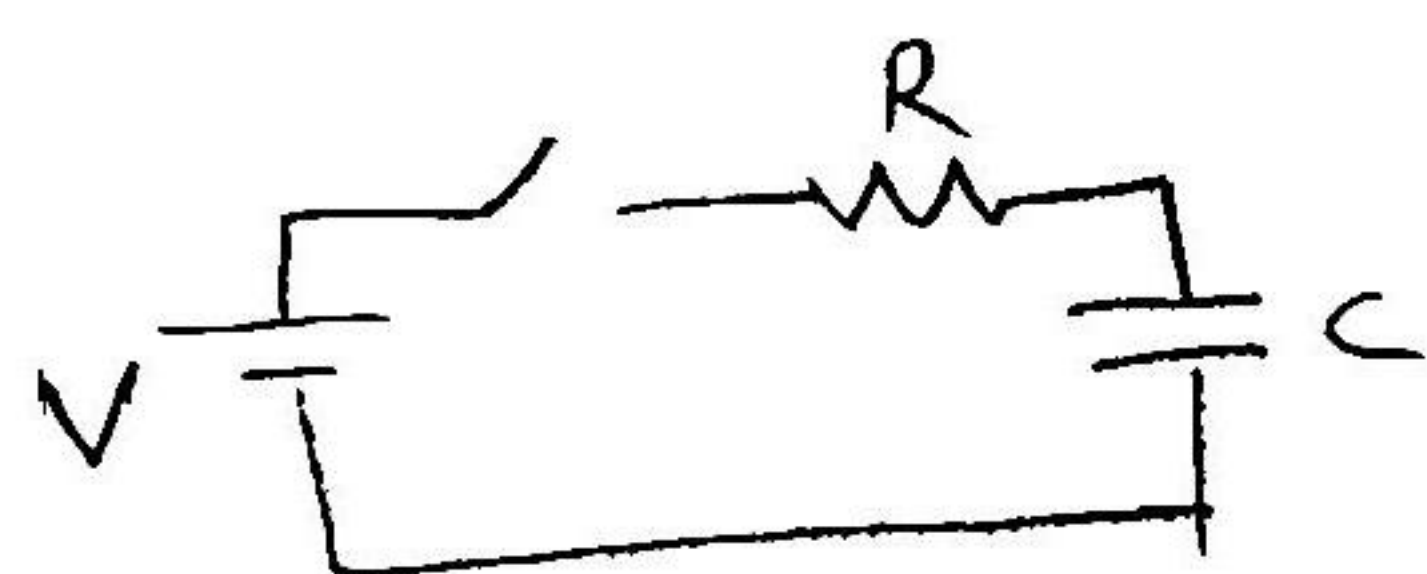
$$V\left(\frac{1}{sL} + \frac{1}{R}\right) = -\frac{I_0}{s} \Rightarrow V = \frac{-\frac{I_0}{s}}{\frac{1}{sL} + \frac{1}{R}} = \frac{-I_0 L}{1 + \frac{sL}{R}} = \frac{-I_0 R}{\frac{R}{L} + s}$$

$$V(s) = -\frac{I_0 R}{s + \frac{R}{L}} \Rightarrow v(t) = -I_0 R e^{-\frac{R}{L}t}$$



$$i_R(t) = \frac{v(t)}{R}, \quad i_L(t) = -i_R(t) = \frac{v(t)}{R} \Rightarrow i_L(t) = I_0 e^{-\frac{R}{L}t}$$

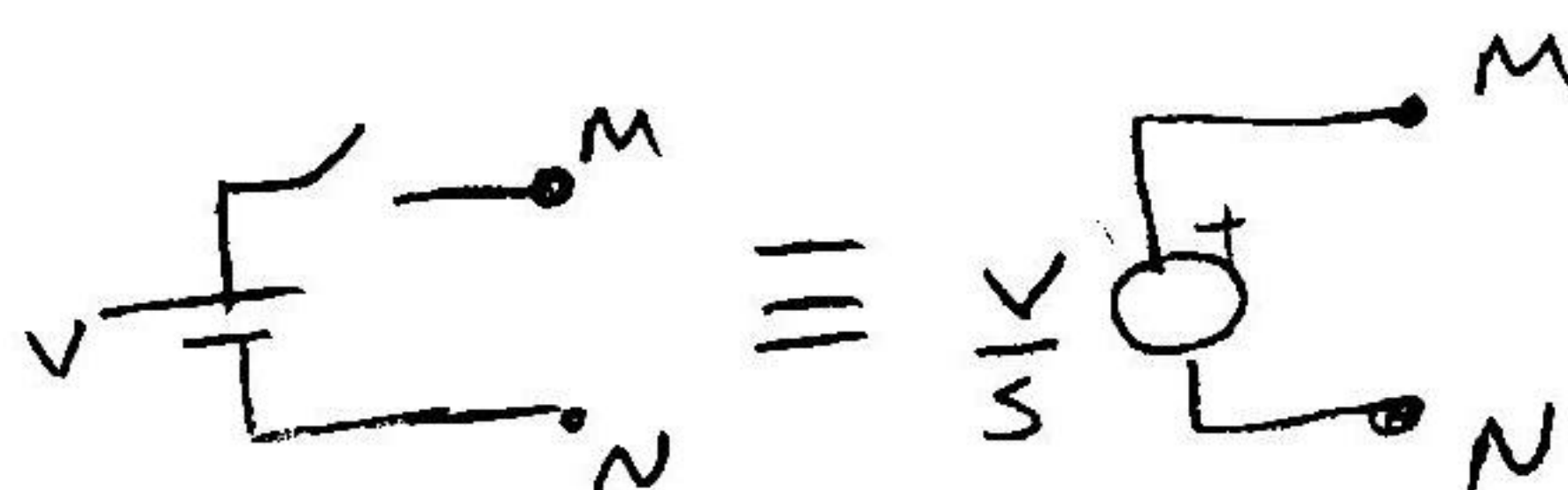
Example problem: calculate $V_c(t)$ if $V_0 = V_c(0) = 20V$



$$R = 2\Omega, \quad C = 0.1F, \quad V = 10V$$

Solution

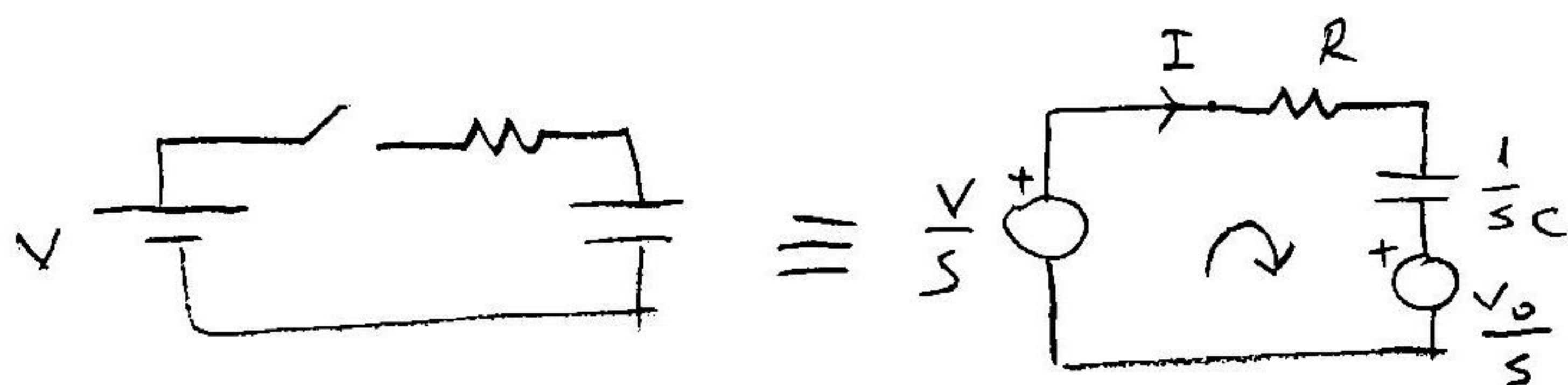
$$+\frac{1}{s} = + \frac{1}{s} \frac{1}{s} \frac{s}{s} = + \frac{1}{s^2} \frac{s}{s}$$



We represent DC source and switch as a step function



$$\Rightarrow \mathcal{L}\{V_{10}(t)\} = \frac{V}{s}$$



$$-\frac{V}{s} + RI + \frac{1}{sC} I + \frac{V_o}{s} = 0$$

$$I(R + \frac{1}{sC}) = \frac{V}{s} - \frac{V_0}{s}$$

$$I = \frac{\frac{V}{s} - \frac{V_0}{s}}{R + \frac{1}{sC}} = \frac{V - V_0}{sR + \frac{1}{C}} = \frac{(V - V_0)/R}{s + \frac{1}{RC}}$$

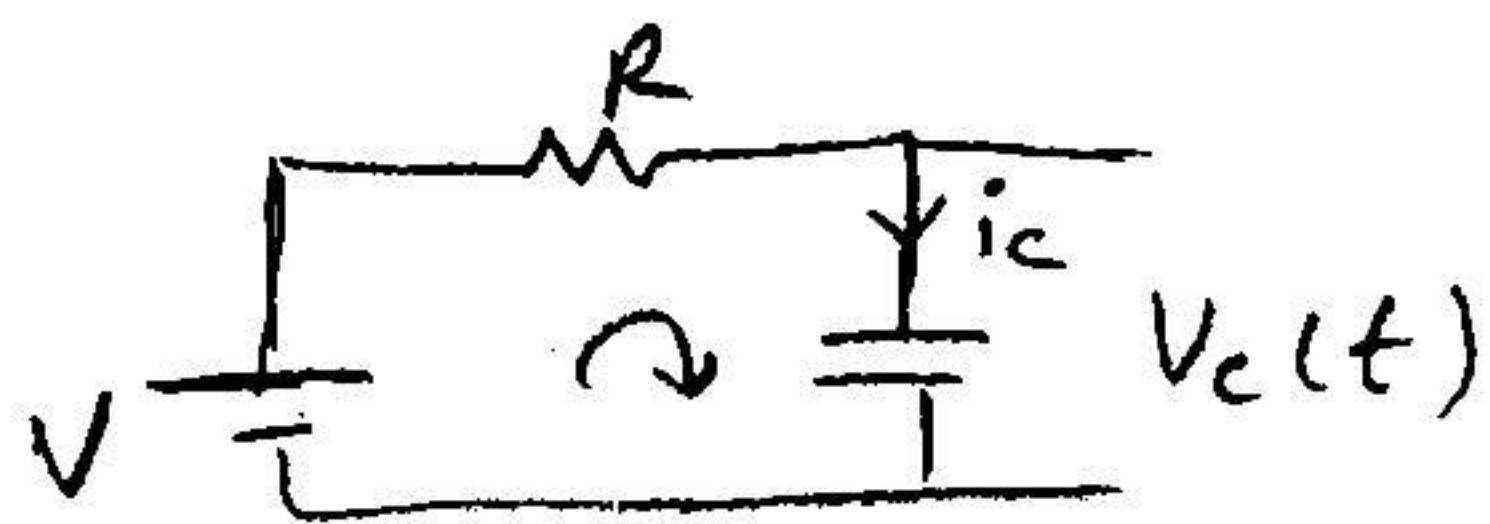
$$I(s) = \frac{(V - V_0)/R}{s + \frac{1}{RC}} \Rightarrow i(t) = \frac{V - V_0}{R} e^{-\frac{1}{RC}t}$$

$$V=10, \quad R=2 \quad C=0.1 \quad V_0=20 \Rightarrow i(t) = \frac{10-20}{2} e^{-\frac{1}{0.2}t} = -5 e^{-5t}$$

Now find $V_c(t)$.

$$i_c(t) = \frac{V-V_0}{R} e^{-\frac{1}{RC}t} \quad 375$$

method 1



$$-V + Ri_c + V_c = 0$$

$$V_c = V - Ri_c$$

$$= V - R \frac{V-V_0}{R} e^{-\frac{1}{RC}t}$$

$$= V - (V-V_0) e^{-\frac{1}{RC}t}$$

$$V_c(t) = 10 - (10-20) e^{-5t}$$

$$= 10 + 10 e^{-5t}$$

method 2

$$i_c(t) = C \frac{dV_c}{dt} \Rightarrow V_c(t) = \frac{1}{C} \int_0^t i_c(\alpha) d\alpha + V_c(0)$$

$$i_c(t) = -5 e^{-5t}$$

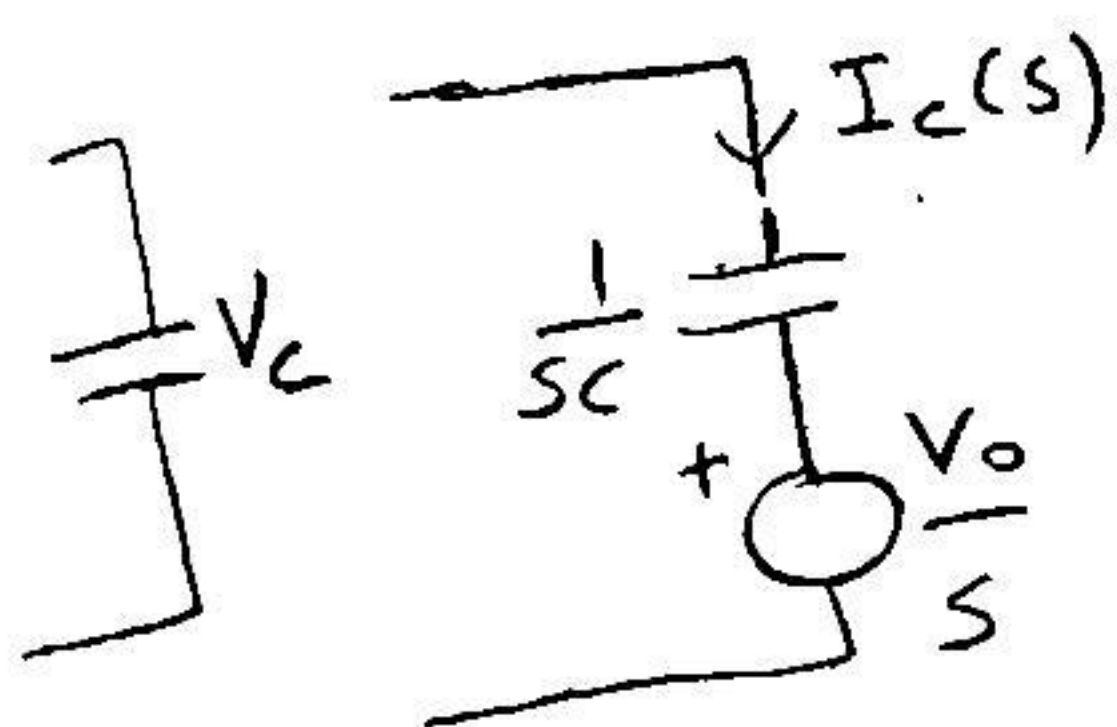
$$i_c(\alpha) = -5 e^{-5\alpha}$$

$$V_c(t) = \frac{1}{0.1} \int_0^t -5 e^{-5\alpha} d\alpha + 20 = \frac{-5}{0.1} \int_0^t e^{-5\alpha} d\alpha + 20$$

$$= \frac{-5}{0.1} \left[\frac{1}{-5} e^{-5\alpha} \right]_0^t + 20 = 10(e^{-5t} - e^0) + 20 = 10 + 10e^{-5t}$$

method 3

$$V_c(s) = \frac{1}{sC} I_c(s) + \frac{V_0}{s} = \frac{1}{sC} \frac{(V-V_0)/R}{s + \frac{1}{RC}} + \frac{V_0}{s}$$



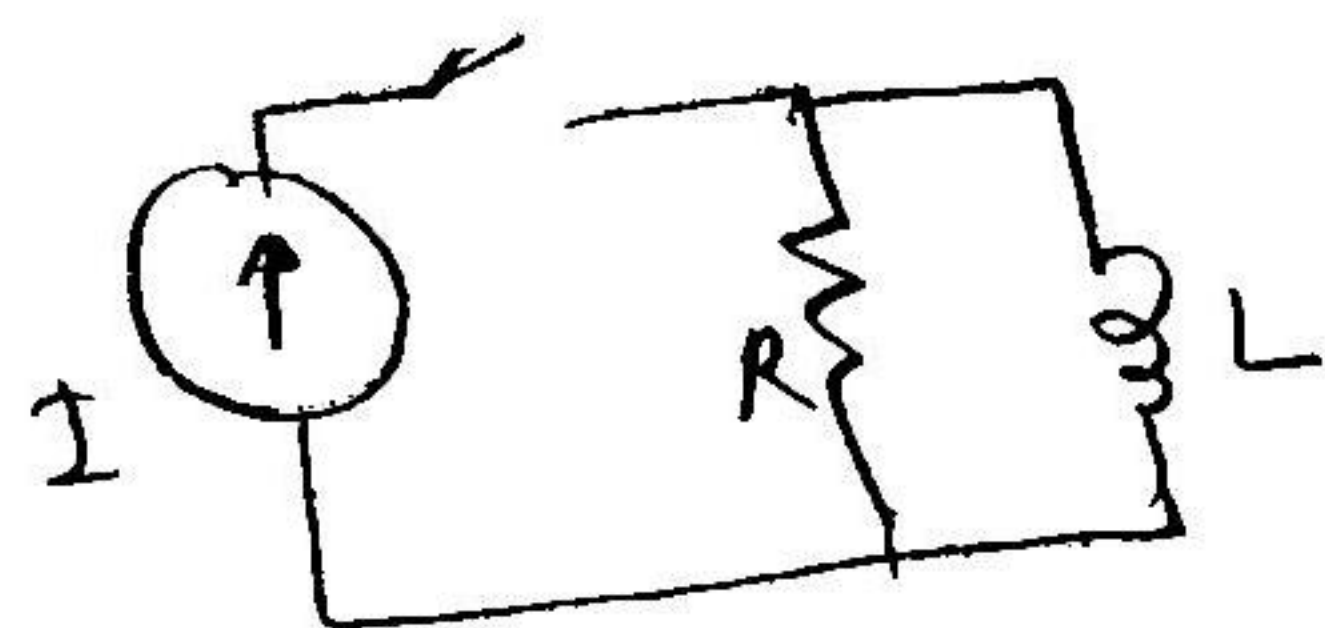
$$= \frac{1}{0.1s} \frac{(10-20)/2}{s + \frac{1}{2 \times 0.1}} + \frac{20}{s} = \frac{-50}{s(s+5)} + \frac{20}{s}$$

$$\frac{1}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5} \Rightarrow A=0.2 \quad B=-0.2$$

$$V_c(s) = -50 \left(\frac{0.2}{s} + \frac{-0.2}{s+5} \right) + \frac{20}{s} = -\frac{10}{s} + \frac{10}{s+5} + \frac{20}{s}$$

$$= \frac{10}{s} + \frac{10}{s+5} \Rightarrow V_c(t) = 10 + 10e^{-5t}$$

Example 74: Calculate $V_L(t)$, $I_L(t)$, $I_L(0) = 0$ 376



$$R = 2, L = 1, I = 10 \text{ A}$$

Solution ^



$$\frac{I}{s} = I_R + I_L \rightarrow \frac{I}{s} = \frac{V}{R} + \frac{V}{sL} \rightarrow \frac{I}{s} = \left(\frac{1}{R} + \frac{1}{sL} \right) V$$

$$V = \frac{\frac{I}{s}}{\frac{1}{R} + \frac{1}{sL}} = \frac{I}{\frac{s}{R} + \frac{1}{L}} = \frac{RI}{s + \frac{R}{L}}$$

$$V(s) = \frac{RI}{s + \frac{R}{L}} \rightarrow V(t) = RI e^{-\frac{R}{L}t}$$

$$V(t) = 2 \times 10 e^{-\frac{2}{1}t} = 20 e^{-2t}$$

$$I_L(s) = \frac{V(s)}{sL} = \frac{1}{sL} \left(\frac{RI}{s + \frac{R}{L}} \right) = \frac{1}{s \times 1} \frac{2 \times 10}{\left(s + \frac{2}{1} \right)} = \frac{20}{s(s+2)}$$

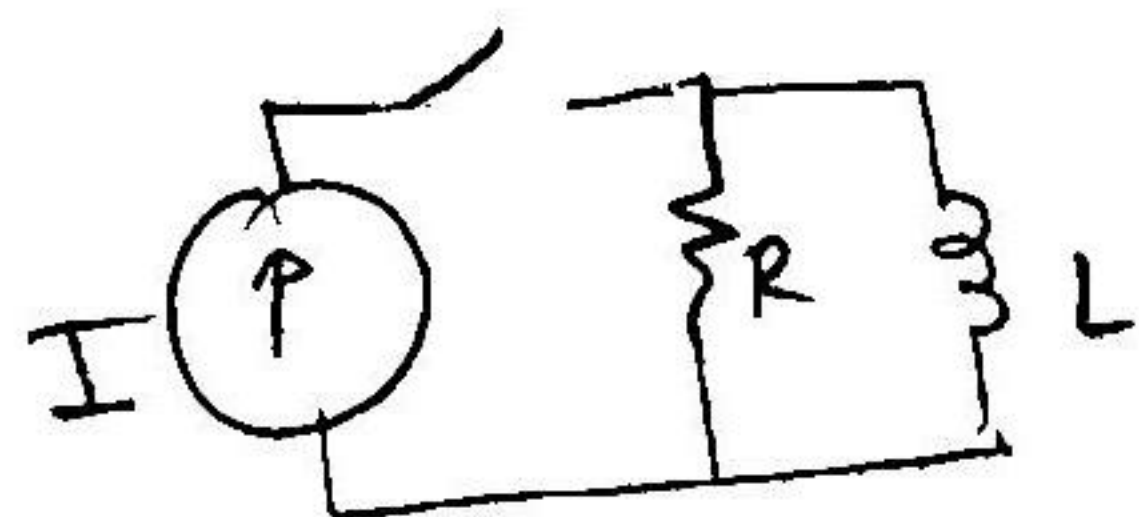
$$= \frac{A}{s} + \frac{B}{s+2}$$

$$A = 10 \quad B = -10$$

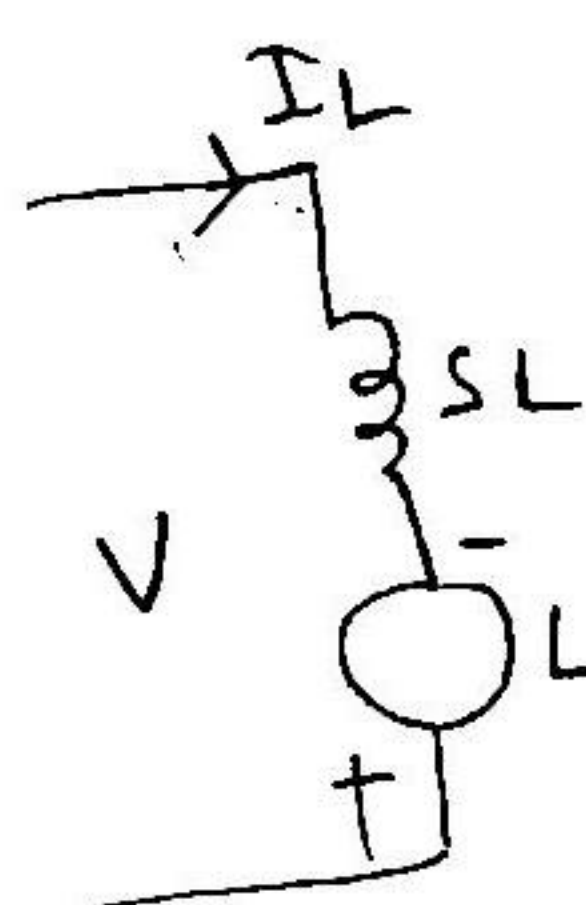
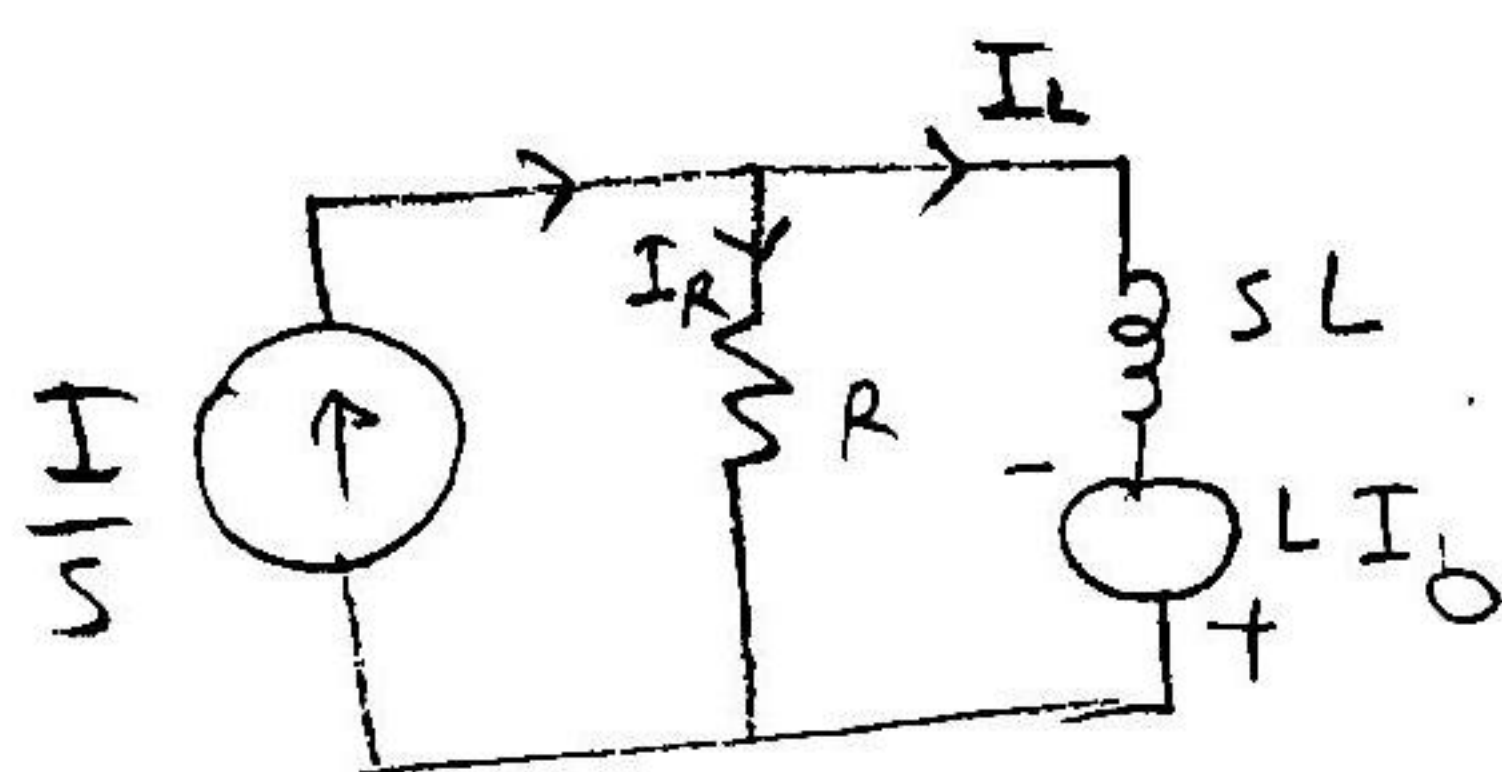
$$I_L(s) = \frac{10}{s} - \frac{10}{s+2} \rightarrow I_L(t) = 10 u(t) - 10 e^{-2t}$$

Example 75 Calculate $V_L(t)$, $I_L(t)$ $I_L(0) = I_0 = 20A$

$$R=2 \quad L=1 \quad I=10A$$



Solution



$$V = sL I_L - L I_0$$

$$I_L = \frac{V + L I_0}{sL}$$

$$\frac{I}{s} = I_R + I_L$$

$$\frac{I}{s} = \frac{V}{R} + \frac{V + L I_0}{sL} \rightarrow \frac{I}{s} - \frac{L I_0}{sL} = \left(\frac{1}{R} + \frac{1}{sL} \right) V$$

$$V = \frac{\frac{I}{s} - \frac{I_0}{s}}{\frac{1}{R} + \frac{1}{sL}} = \frac{I - I_0}{\frac{s}{R} + \frac{1}{L}} = \frac{R(I - I_0)}{s + \frac{R}{L}} = \frac{2(10 - 20)}{s + \frac{2}{1}}$$

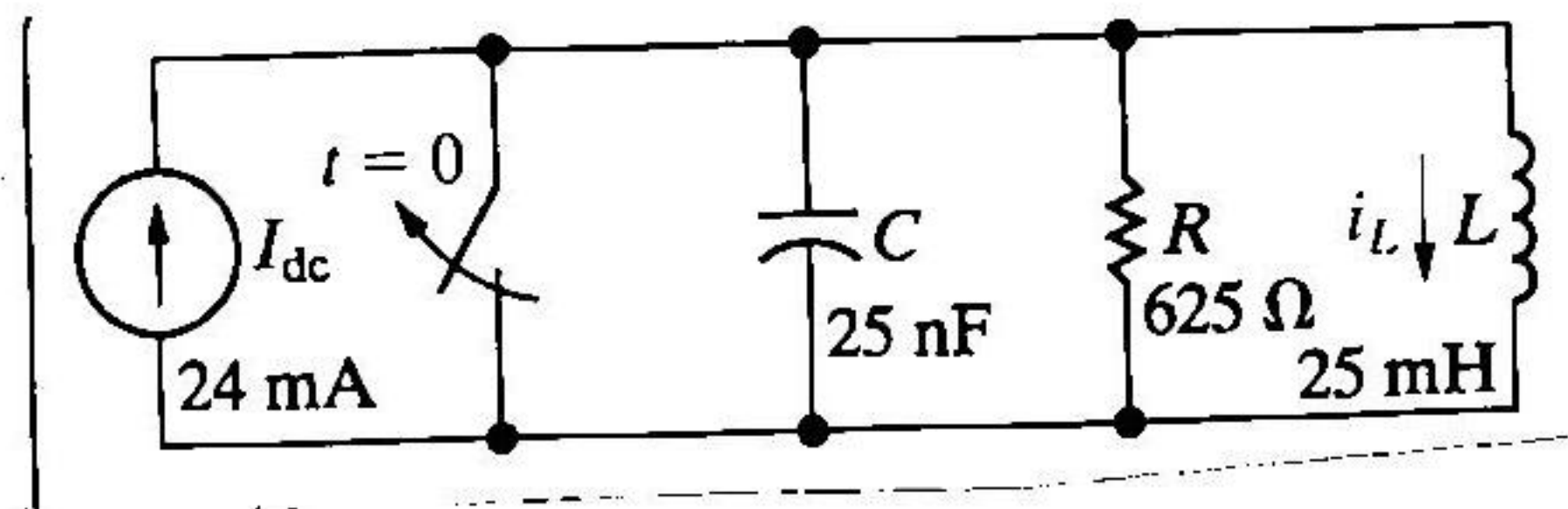
$$V = \frac{-20}{s + 2} \rightarrow V(t) = -20 e^{-2t}$$

$$I_L(s) = \frac{V + L I_0}{sL} = \frac{-\frac{20}{s+2} + L I_0}{sL} = \frac{-\frac{20}{s+2} + 1 \times 20}{1 \times s} = \frac{20}{s} - \frac{20}{s(s+2)}$$

$$\frac{20}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2} \quad A=10 \quad B=-10$$

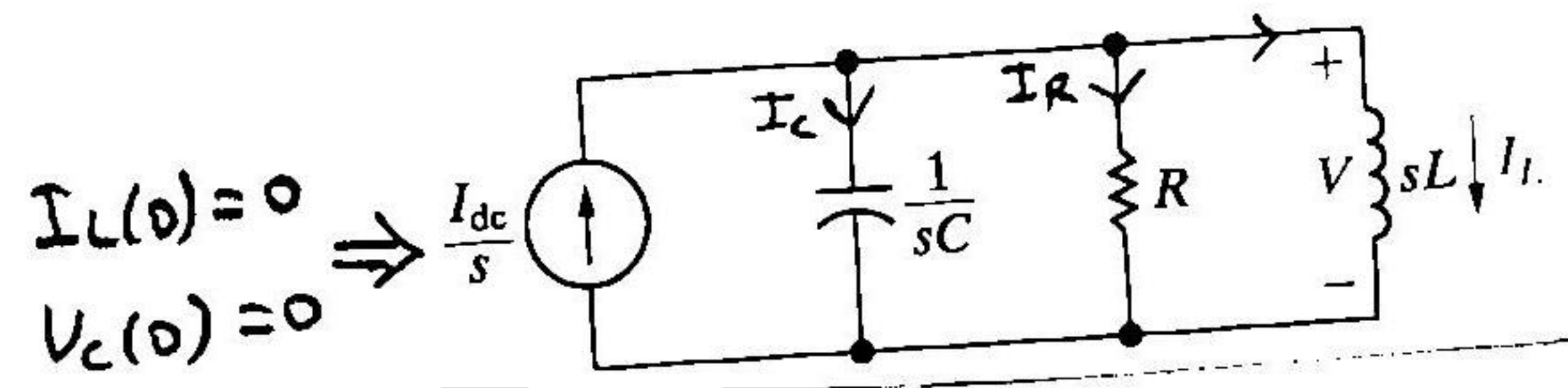
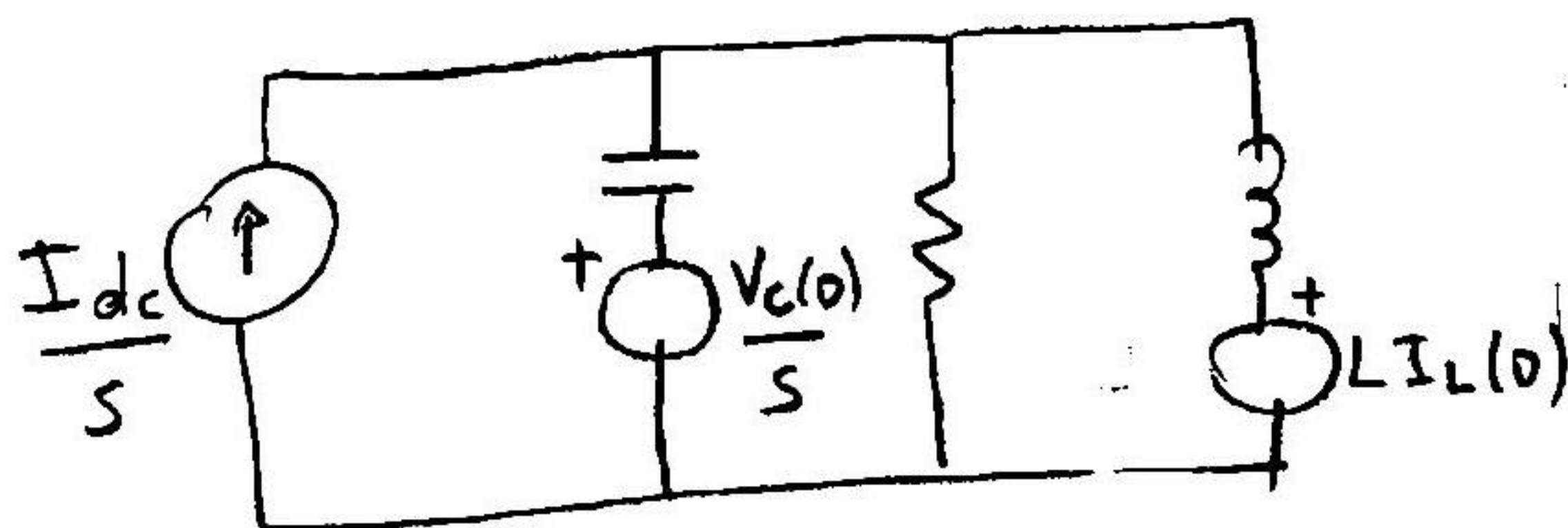
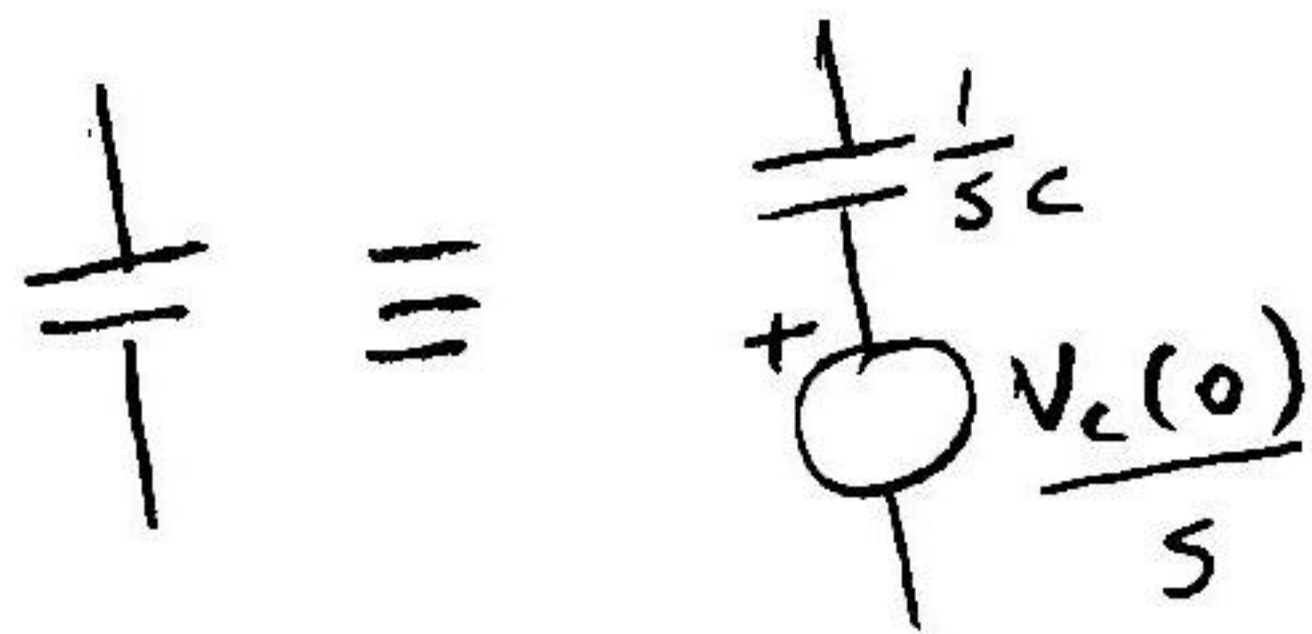
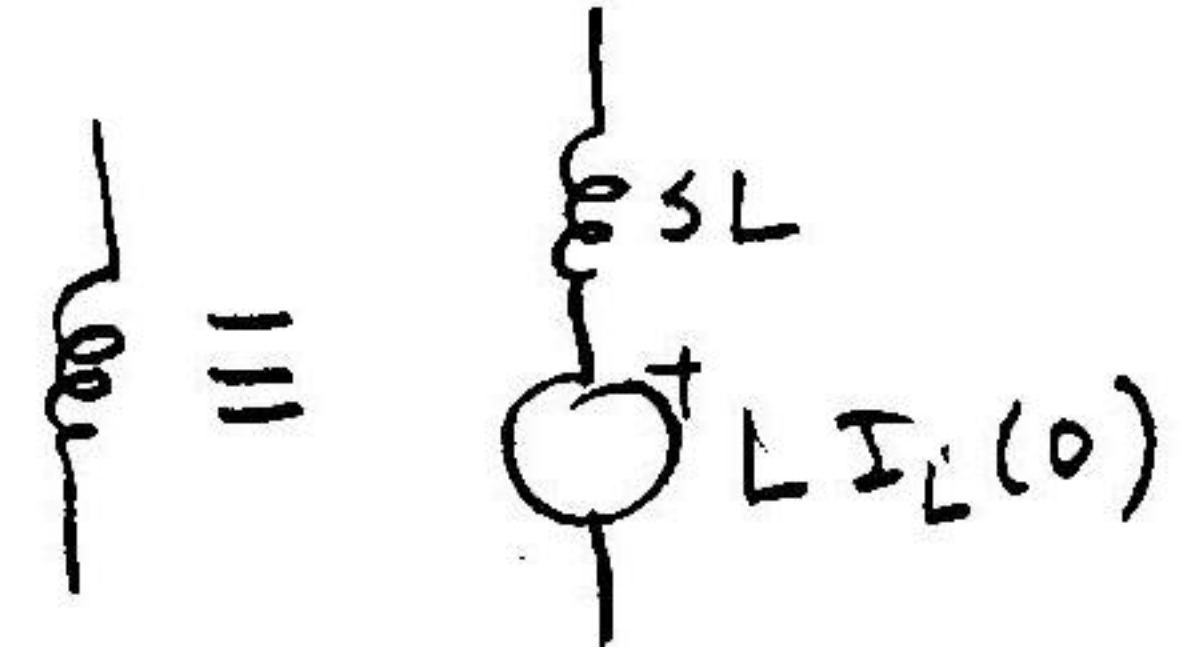
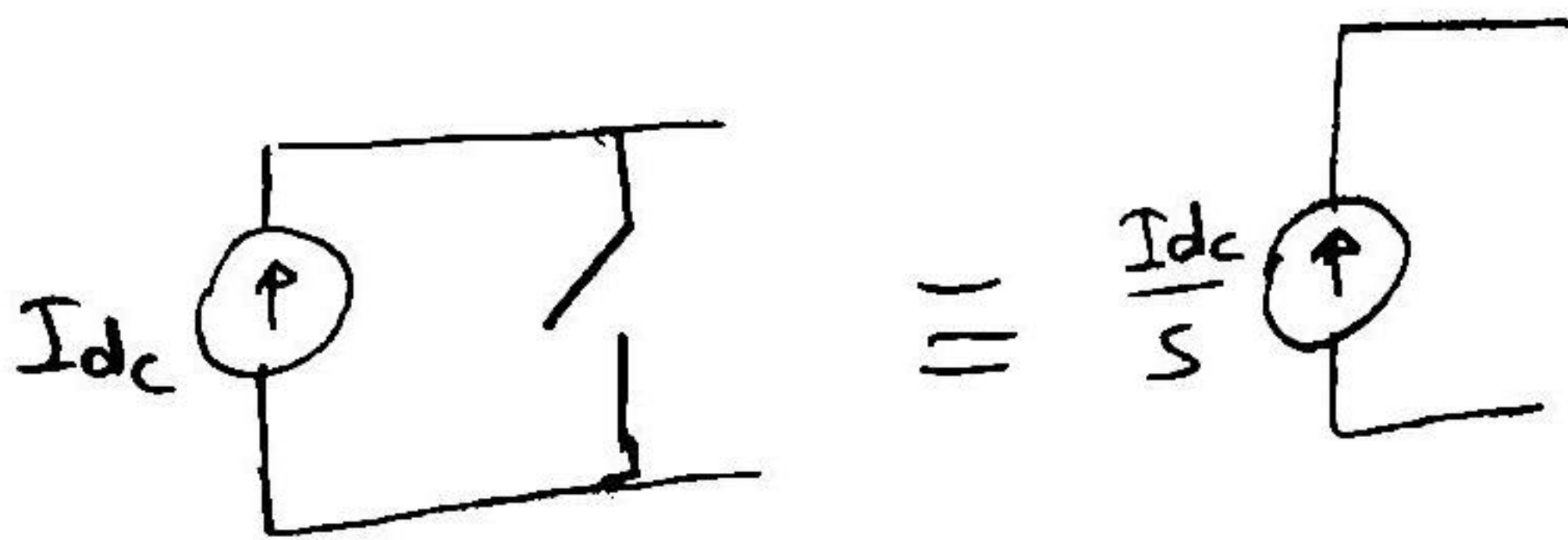
$$I_L(s) = \frac{20}{s} - \frac{20}{s(s+2)} = \frac{20}{s} - \left(\frac{10}{s} - \frac{10}{s+2} \right) = \frac{10}{s} + \frac{10}{s+2}$$

$$I_L(t) = 10 u(t) + 10 e^{-2t}$$



Calculate currents and voltage if $I_L(0) = 0$ $V_C(0) = 0$

solution



$$\frac{I_{dc}}{s} = I_C + I_R + I_L$$

$$\frac{I_{dc}}{s} = \frac{V}{\frac{1}{sC}} + \frac{V}{R} + \frac{V}{sL} \Rightarrow \frac{I_{dc}}{s} = \left(sC + \frac{1}{R} + \frac{1}{sL} \right) V$$

$$V = \frac{I_{dc}}{s} \frac{1}{sC + \frac{1}{R} + \frac{1}{sL}} = \frac{I_{dc}}{s} \frac{sL}{s^2 LC + s \frac{L}{R} + 1} = \frac{I_{dc}/C}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

$$V = \frac{24 \times 10^{-3} / 25 \times 10^{-9}}{s^2 + \frac{1}{625 \times 25 \times 10^{-9}} s + \frac{1}{25 \times 10^{-3} \times 25 \times 10^{-9}}} = \frac{960000}{s^2 + 64000s + 16 \times 10^8}$$

$$V(s) = \frac{960000}{s^2 + 64000s + 16 \times 10^8}$$

$$s^2 + 64000s + 16 \times 10^8 = 0 \quad \begin{matrix} \rightarrow s_1 = -32000 + 24000j \\ \rightarrow s_2 = -32000 - 24000j \end{matrix}$$

$$s^2 + 64000s + 16 \times 10^8 = (s + 32000)^2 + 24000^2$$

$$V(s) = \frac{960000}{24000} \cdot \frac{24000}{(s + 32000)^2 + 24000^2}$$

$$\mathcal{L}^{-1} \left\{ \frac{b}{(s+a)^2 + b^2} \right\} = e^{-at} \sin bt$$

$$v(t) = \frac{960000}{24000} \cdot e^{-32000t} \sin 24000t$$

Now calculate $i_L(t)$ and $i_C(t)$

$$\begin{matrix} \downarrow I \\ sL \uparrow V \end{matrix} \quad I_L = \frac{V(s)}{sL} = \frac{V(s)}{25 \times 10^{-3} s} = \frac{960000}{25 \times 10^{-3} s} \cdot \frac{1}{s^2 + 64000s + 16 \times 10^8}$$

$$I_L(s) = \frac{38400000}{s(s^2 + 64000s + 16 \times 10^8)} = \frac{A}{s} + \frac{B}{s_1} + \frac{C}{s_2}$$

$$\text{Here } s_1 = -32000 + 24000j \quad s_2 = -32000 - 24000j$$

$$A = s \cdot \frac{38400000}{s(s^2 + 64000s + 16 \times 10^8)} \bigg|_{s=0} = \frac{38400000}{16 \times 10^8} = 0.024$$

$$B = (s - s_1) \frac{38400000}{s(s - s_1)(s - s_2)} \bigg|_{s=s_1} = \frac{38400000}{s_1(s_1 - s_2)}$$

$$= \frac{38400000}{(-32000 + 24000j) [-32000 + 24000j - (-32000 - 24000j)]} = -0.012 + 0.016j$$

$$C = B^* = (-0.012 + 0.016j)^* = -0.012 - 0.016j \quad 380$$

$$I_L(s) = \frac{0.024}{s} + \frac{-0.012 + 0.016j}{s - (-32000 + 24000j)} + \frac{-0.012 - 0.016j}{s - (-32000 - 24000j)}$$

$$\mathcal{L}^{-1} \left\{ \frac{a+bi}{s-(x+iy)} + \frac{a-bi}{s-(x-iy)} \right\} = e^{xt} [2a \cos yt - 2b \sin yt]$$

here $a = -0.012$ $b = 0.016$ $x = -32000$ $y = 24000$

$$i_L(t) = 0.024 u(t) + e^{-32000t} (2 \times (-0.012) \cos 24000t - 2 \times (0.016) \sin 24000t)$$

$$i_L(t) = 0.024 u(t) + e^{-32000t} (-0.024 \cos 24000t - 0.032 \sin 24000t)$$

$$i_C(t) = i_{dc} - [i_R(t) + i_L(t)]$$

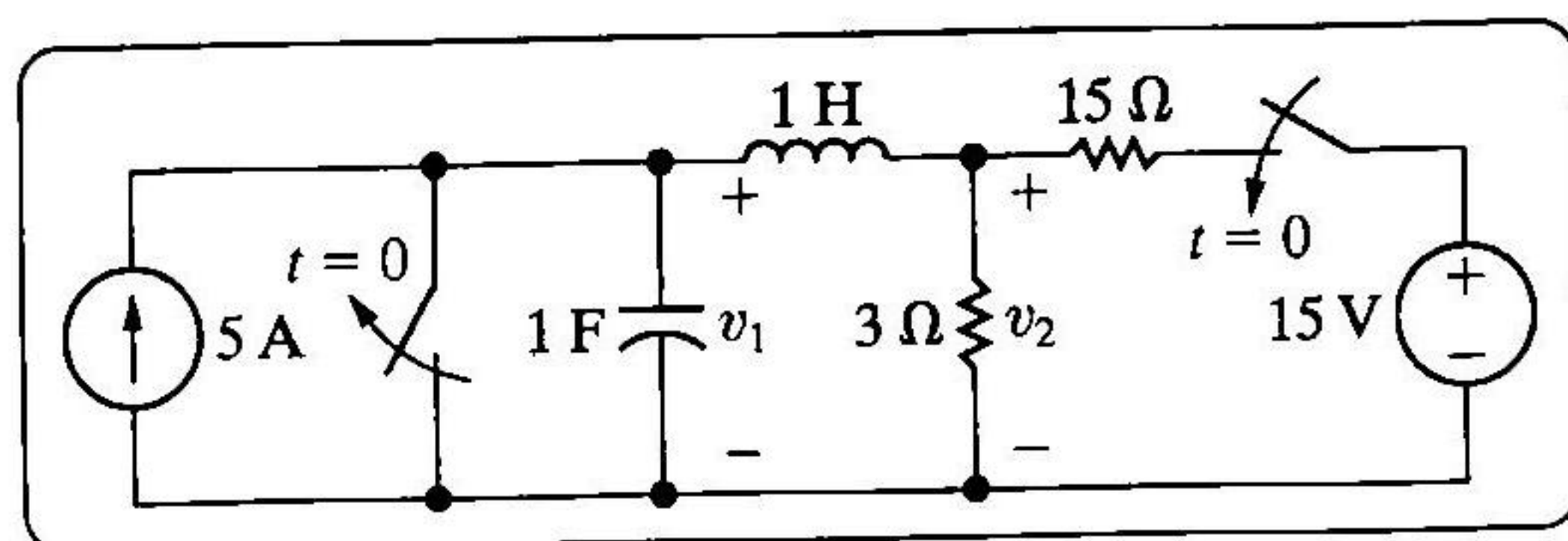
$$i_R(t) = \frac{v(t)}{R} \quad i_{dc} = 24$$

replace $v(t)$, i_{dc} , $i_L(t)$, $i_R(t)$ and calculate $i_C(t)$

13.5

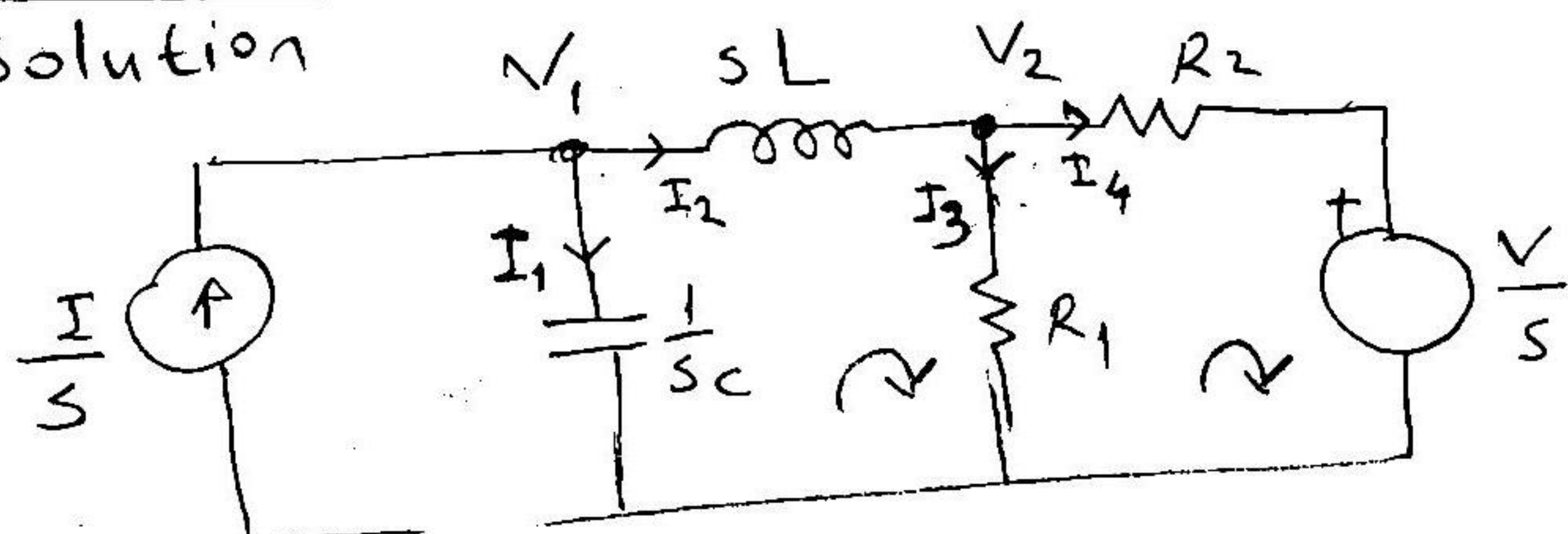
The dc current and voltage sources are applied simultaneously to the circuit shown. No energy is stored in the circuit at the instant of application.

- Derive the s -domain expressions for V_1 and V_2 .
- For $t > 0$, derive the time-domain expressions for v_1 and v_2 .
- Calculate $v_1(0^+)$ and $v_2(0^+)$.
- Compute the steady-state values of v_1 and v_2 .



ANSWER: (a) $V_1 = [5(s+3)]/[s(s+0.5)(s+2)]$,
 $V_2 = [2.5(s^2+6)]/[s(s+0.5)(s+2)]$;
 (b) $v_1 = (15 - \frac{50}{3}e^{-0.5t} + \frac{5}{3}e^{-2t})u(t)$ V,
 $v_2 = (15 - \frac{125}{6}e^{-0.5t} + \frac{25}{3}e^{-2t})u(t)$ V;
 (c) $v_1(0^+) = 0$, $v_2(0^+) = 2.5$ V;
 (d) $v_1 = v_2 = 15$ V.

Solution



$$I = 5 \quad V = 15$$

$$C = 1 \quad L = 1 \quad R_1 = 3$$

$$R_2 = 15$$

node equations $\frac{I}{s} = I_1 + I_2 \quad (1)$

$$I_2 = I_3 + I_4 \quad (2)$$

loop equations $-V_1 + sL I_2 + V_2 = 0 \quad (3)$

$$-V_2 + R_2 I_4 + \frac{V}{s} = 0 \quad (4)$$

We try to solve V_1 and V_2

$$(3) \rightarrow I_2 = \frac{V_1 - V_2}{sL} \quad (5)$$

$$(4) \rightarrow I_4 = \frac{V_2 - \frac{V}{s}}{R_2} \quad (6)$$

$$(1), (5) \rightarrow \frac{I}{s} = \frac{V_1}{\frac{1}{sC}} + \frac{V_1 - V_2}{sL}$$

$$\frac{I}{s} = sC V_1 + \frac{V_1}{sL} - \frac{V_2}{sL}$$

$$\frac{I}{s} = \underbrace{\left(sC + \frac{1}{sL}\right)}_A V_1 - \frac{V_2}{sL} \Rightarrow \frac{I}{s} = A V_1 - B V_2 \quad (7)$$

$$B = \frac{1}{sL}$$

$$(2) \rightarrow I_2 = I_3 + I_4$$

$$(5), (6) \rightarrow \frac{V_1 - V_2}{sL} = \frac{V_2}{R_1} + \frac{V_2 - \frac{V}{s}}{R_2} \rightarrow \frac{V_1}{sL} - \frac{V_2}{sL} = \frac{V_2}{R_1} + \frac{V_2}{R_2} - \frac{V}{sR_2}$$

$$\left(\begin{array}{l} B = \frac{1}{sL} \\ D = \frac{1}{sR_2} \end{array} \right) \quad BV_1 - BV_2 = \frac{V_2}{R_1} + \frac{V_2}{R_2} - DV$$

$$BV_1 = \underbrace{\left(B + \frac{1}{R_1} + \frac{1}{R_2} \right)}_E V_2 + DV$$

$$BV_1 = E V_2 + DV \quad (8)$$

$$V_1 = \frac{E}{B} V_2 + \frac{D}{B} V \quad (9)$$

$$(7) \rightarrow \frac{I}{s} = A V_1 - B V_2$$

$$\frac{I}{s} = A \left(\frac{E}{B} V_2 + \frac{D}{B} V \right) - B V_2$$

$$\frac{I}{s} - \frac{AD}{B} V = \left(\frac{AE}{B} - B \right) V_2$$

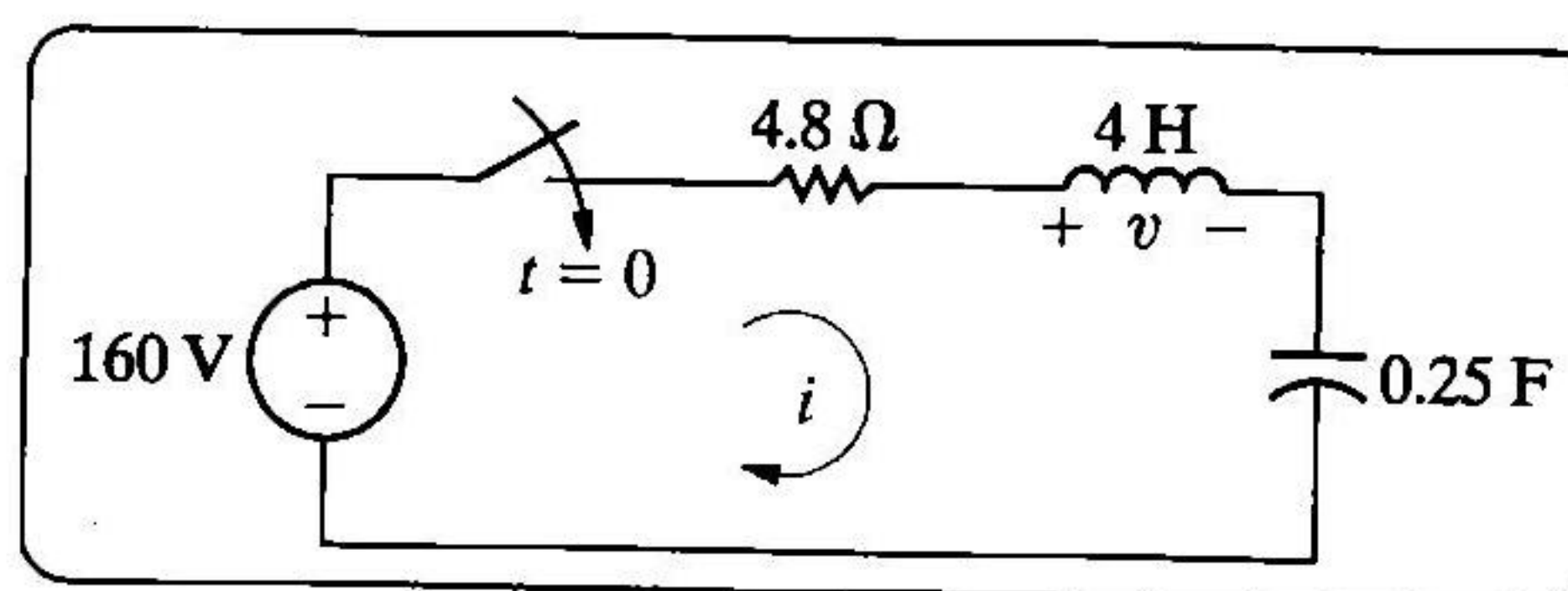
$$\frac{\frac{I}{s} - \frac{AD}{B} V}{\frac{AE}{B} - B} = V_2 \Rightarrow V_2 = \frac{IB - SAD V}{AES - B^2}$$

$$V_2 = \frac{I \frac{1}{sL} - s \left(sC + \frac{1}{sL} \right) \frac{1}{sR_2} V}{\left(sC + \frac{1}{sL} \right) \left(\frac{1}{sL} + \frac{1}{R_1} + \frac{1}{R_2} \right) s - \frac{1}{(sL)^2}}$$

13.4

The energy stored in the circuit shown is zero at the time when the switch is closed.

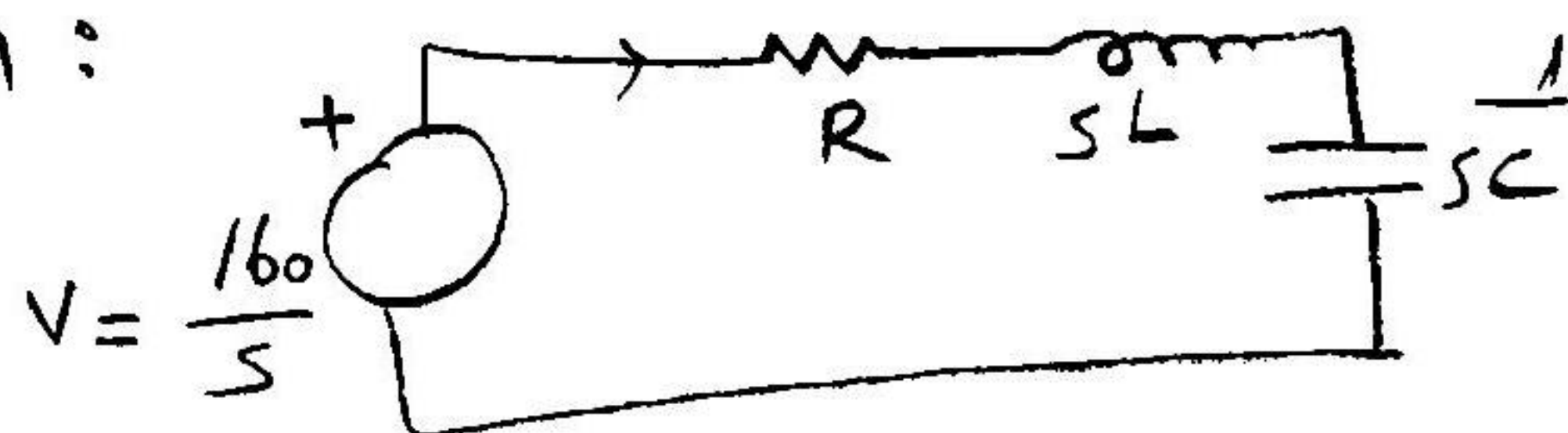
- Find the s -domain expression for I .
- Find the time-domain expression for i when $t > 0$.
- Find the s -domain expression for V .
- Find the time-domain expression for v when $t > 0$.



ANSWER: (a) $I = 40/(s^2 + 1.2s + 1)$;
 (b) $i = (50e^{-0.6t} \sin 0.8t)u(t)$ A;
 (c) $V = 160s/(s^2 + 1.2s + 1)$;
 (d) $v = [200e^{-0.6t} \cos(0.8t + 36.87^\circ)]u(t)$ V.

NOTE ◆ Also try Chapter Problems 13.15 and 13.16.

Solution:



$$I(s) = \frac{V}{R + sL + \frac{1}{sC}} = \frac{sC V}{sC R + s^2 LC + 1} = \frac{sC \frac{160}{s}}{LC s^2 + RC s + 1}$$

$$= \frac{160 C / LC}{s^2 + \frac{R}{L} s + \frac{1}{LC}} = \frac{160/4}{s^2 + \frac{4.8}{4} s + \frac{1}{4 \times 0.25}}$$

$$= \frac{40}{s^2 + 1.2s + 1}$$

$$V(s) = sL I(s) = s \times 4 \frac{40}{s^2 + 1.2s + 1} = \frac{160s}{s^2 + 1.2s + 1}$$

$$s^2 + 1.2s + 1 = 0 \quad s_{1,2} = \frac{-1.2 \pm \sqrt{1.2^2 - 4}}{2} = -0.6 \pm 0.8j$$

$$s^2 + 1.2s + 1 = (s + 0.6)^2 + 0.8^2$$

$$I(s) = \frac{40}{(s + 0.6)^2 + 0.8^2} = \frac{40}{0.8} \frac{0.8}{(s + 0.6)^2 + 0.8^2} \rightarrow i(t) = 50e^{-0.6t} \sin 0.8t$$

$$V(s) = \frac{160s}{(s + 0.6)^2 + 0.8^2} = 160 \left(\frac{s + 0.6}{(s + 0.6)^2 + 0.8^2} - \frac{0.6}{(s + 0.6)^2 + 0.8^2} \right)$$

$$V(s) = 160 \left(\frac{s+0.6}{(s+0.6)^2 + 0.8^2} - \frac{0.6}{0.8} \frac{0.8}{(s+0.6)^2 + 0.8^2} \right)$$

$$v(t) = 160 \left[e^{-0.6t} \cos 0.8t - 0.75 e^{-0.6t} \sin 0.8t \right]$$

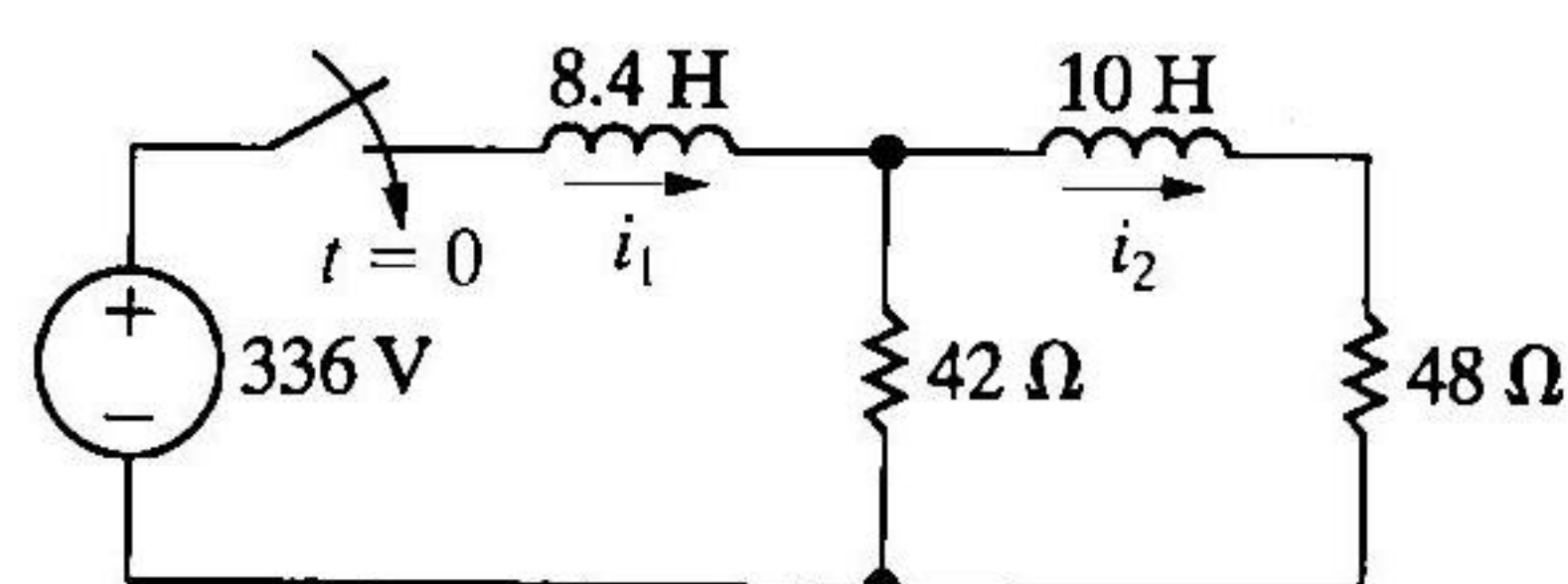
$$= e^{-0.6t} [160 \cos 0.8t - 120 \sin 0.8t]$$

$$A \cos x + B \sin x = \sqrt{A^2 + B^2} \cos(x - \tan^{-1} \frac{B}{A})$$

$$v(t) = e^{-0.6t} \sqrt{160^2 + 120^2} \cos(0.8t - \tan^{-1} \frac{-120}{160})$$

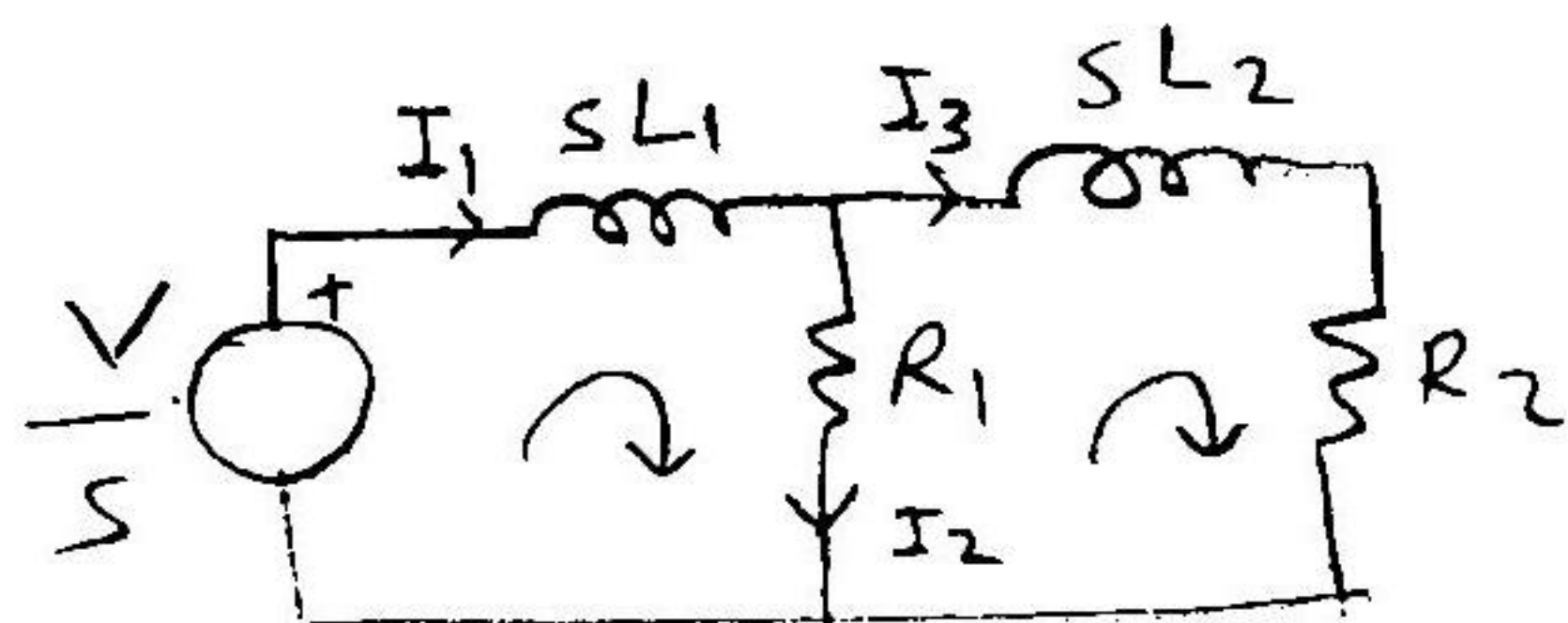
$$= 200 e^{-0.6t} \cos(0.8t + 36.8)$$

The Step Response of a Multiple Mesh Circuit



Calculate currents and voltages. initial currents are zero

Solution



$$-\frac{V}{s} + sL_1 I_1 + R_1 I_2 = 0 \quad (1)$$

$$-R_1 I_2 + (sL_2 + R_2) I_3 = 0 \quad (2)$$

$$I_1 = I_2 + I_3 \quad (3)$$

3 equations 3 unknowns

$$(2) \rightarrow I_2 = \frac{sL_2 + R_2}{R_1} I_3 = M I_3 \quad (4)$$

$$(1) \text{ and } (3) \rightarrow -\frac{V}{s} + sL_1 (I_2 + I_3) + R_1 I_2 = 0 \quad (5)$$

$$-\frac{V}{s} + sL_1 (M I_3 + I_3) + R_1 M I_3 = 0$$

$$(sL_1 M + sL_1 + R_1 M) I_3 = \frac{V}{s}$$

$$I_3 = \frac{V}{s} \frac{1}{sL_1 M + sL_1 + R_1 M} = \frac{V}{s} \frac{1}{sL_1 \frac{sL_2 + R_2}{R_1} + sL_1 + R_1 \frac{sL_2 + R_2}{R_1}}$$

$$= \frac{V}{s} \frac{1}{\frac{L_1 L_2}{R_1} s^2 + (L_1 \frac{R_2}{R_1} + L_1 + L_2) s + R_2}$$

$$V = 336 \quad L_1 = 8.4 \quad L_2 = 10 \quad R_1 = 42 \quad R_2 = 48$$

$$I_3 = \frac{336}{s} \frac{1}{2s^2 + 28s + 48} = \frac{336}{s} \frac{1/2}{s^2 + 14s + 24}$$

$$I_3(s) = \frac{168}{s(s^2 + 14s + 24)}$$

$$I_2(s) = M I_3 = \frac{sL_2 + R_2}{R_1} \frac{168}{s(s^2 + 14s + 24)} = \frac{40s + 192}{s(s^2 + 14s + 24)}$$

$$I_1 = I_2 + I_3 = \Rightarrow i_1(t) = 15 - 14e^{-2t} - e^{-12t}$$

$$V_{L1} = sL_1 I_1 =$$

$$V_{R1} = R_1 I_2 =$$

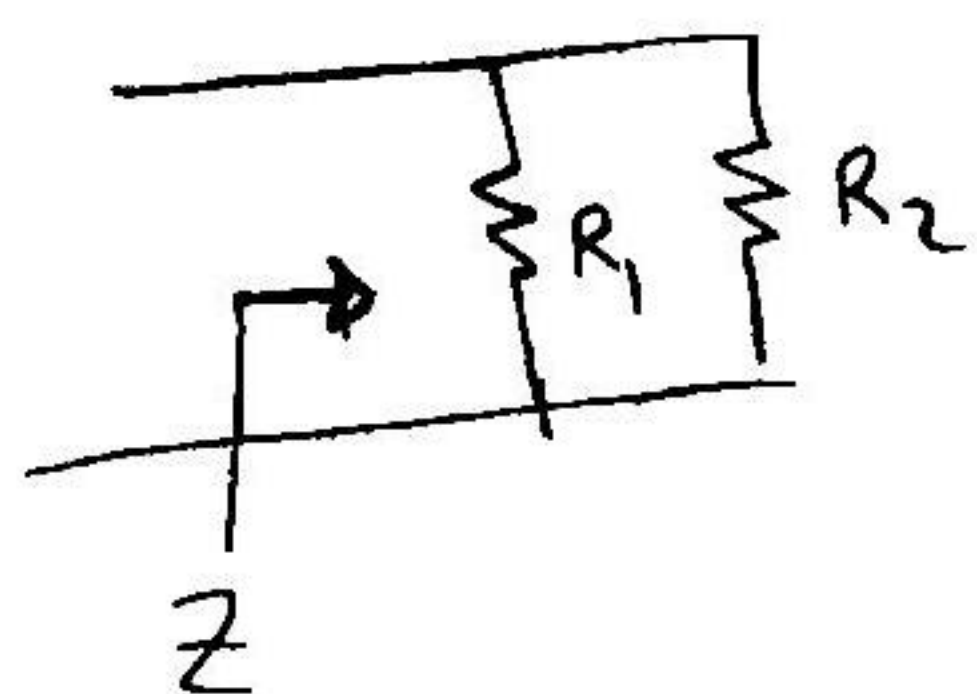
$$V_{L2} = sL_2 I_3 =$$

$$V_{R2} = R_2 I_3 =$$

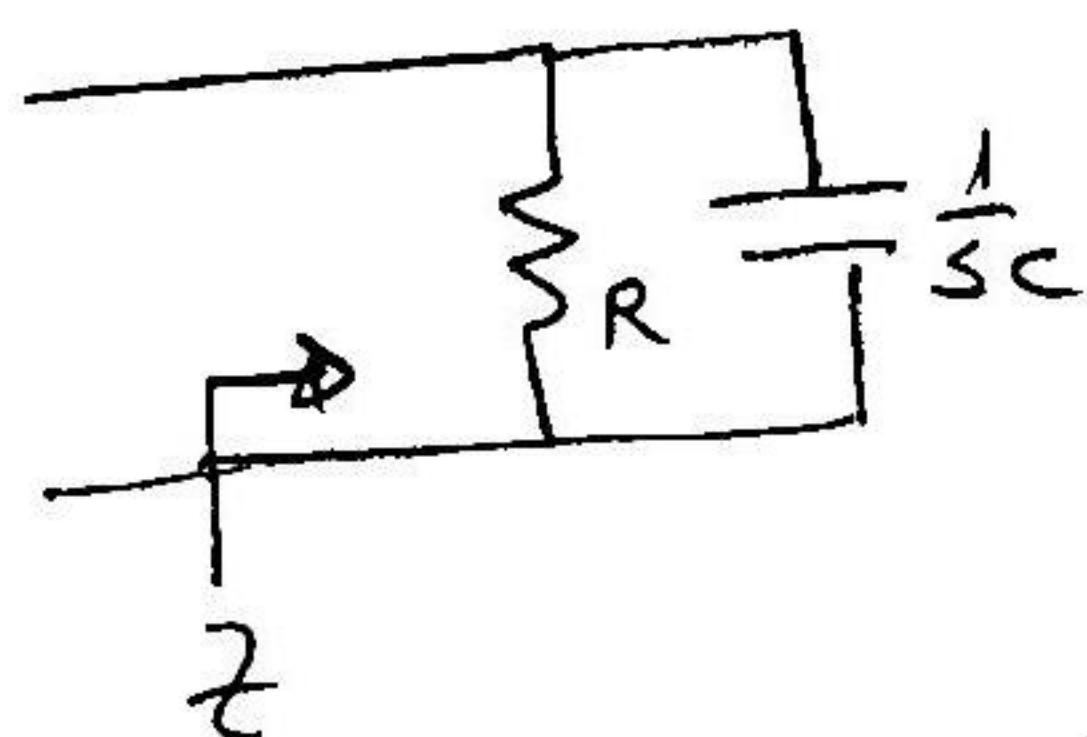
$$i_3(t) = 7 - 8.4 e^{-2t} + 1.4 e^{-12t}$$

$$i_2(t) = 8 - 5.6 e^{-2t} - 2.4 e^{-12t}$$

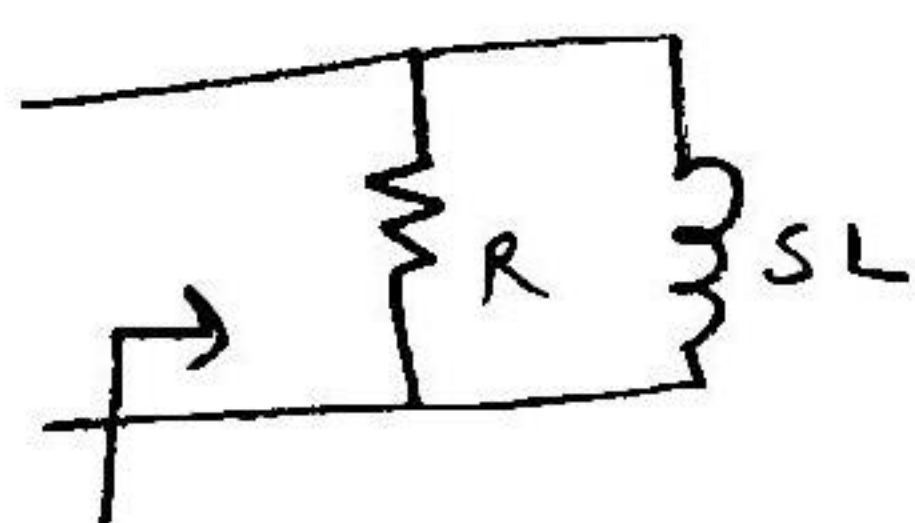
Calculate the impedances



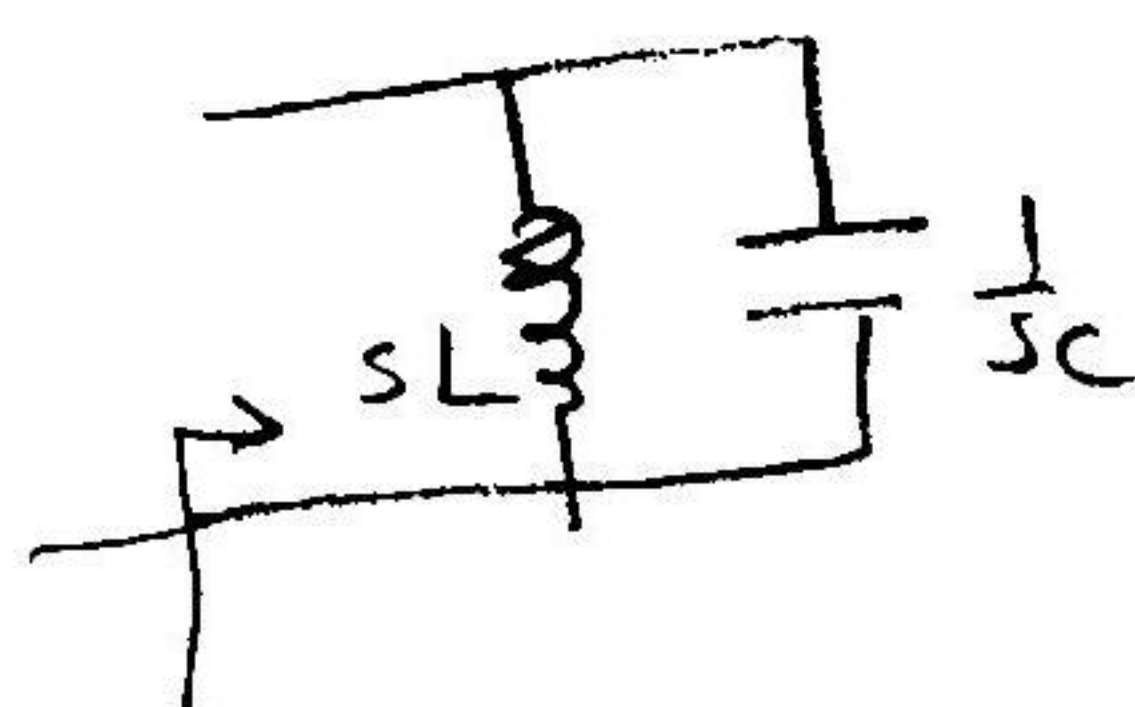
$$Z = \frac{R_1 R_2}{R_1 + R_2}$$



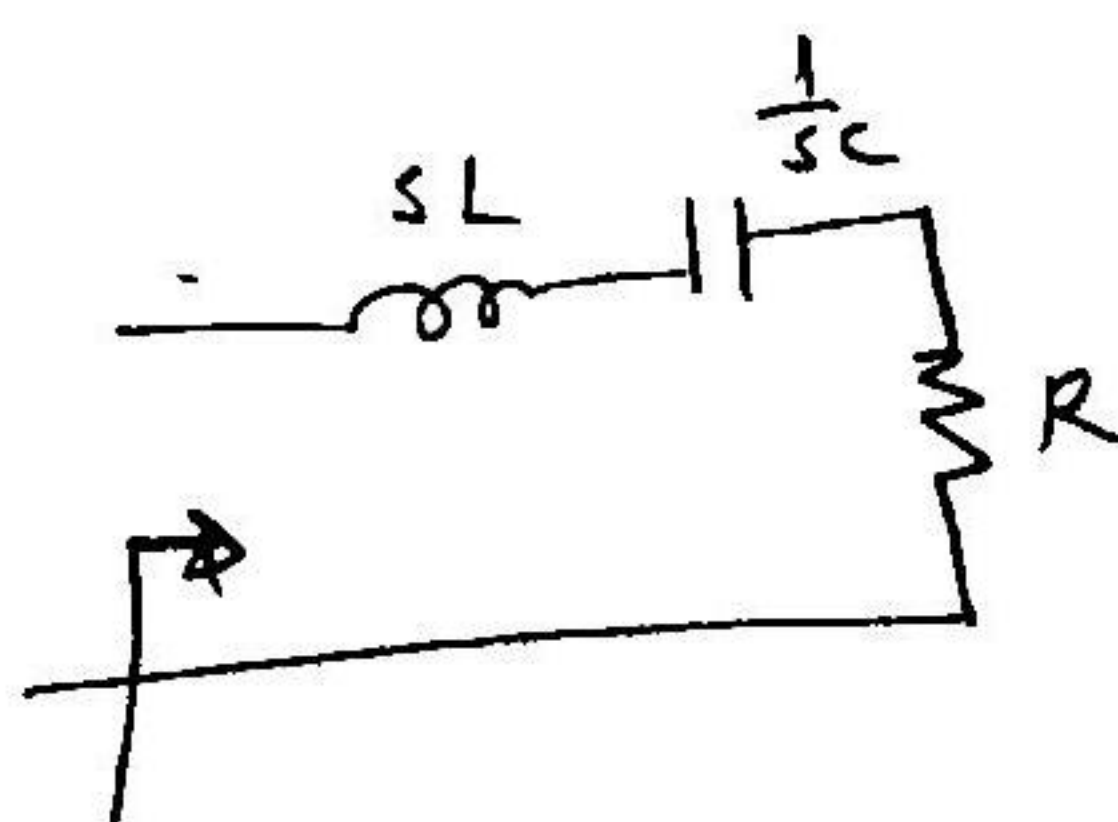
$$Z = \frac{R \cdot \frac{1}{sC}}{R + \frac{1}{sC}} = \frac{R}{sCR + 1}$$



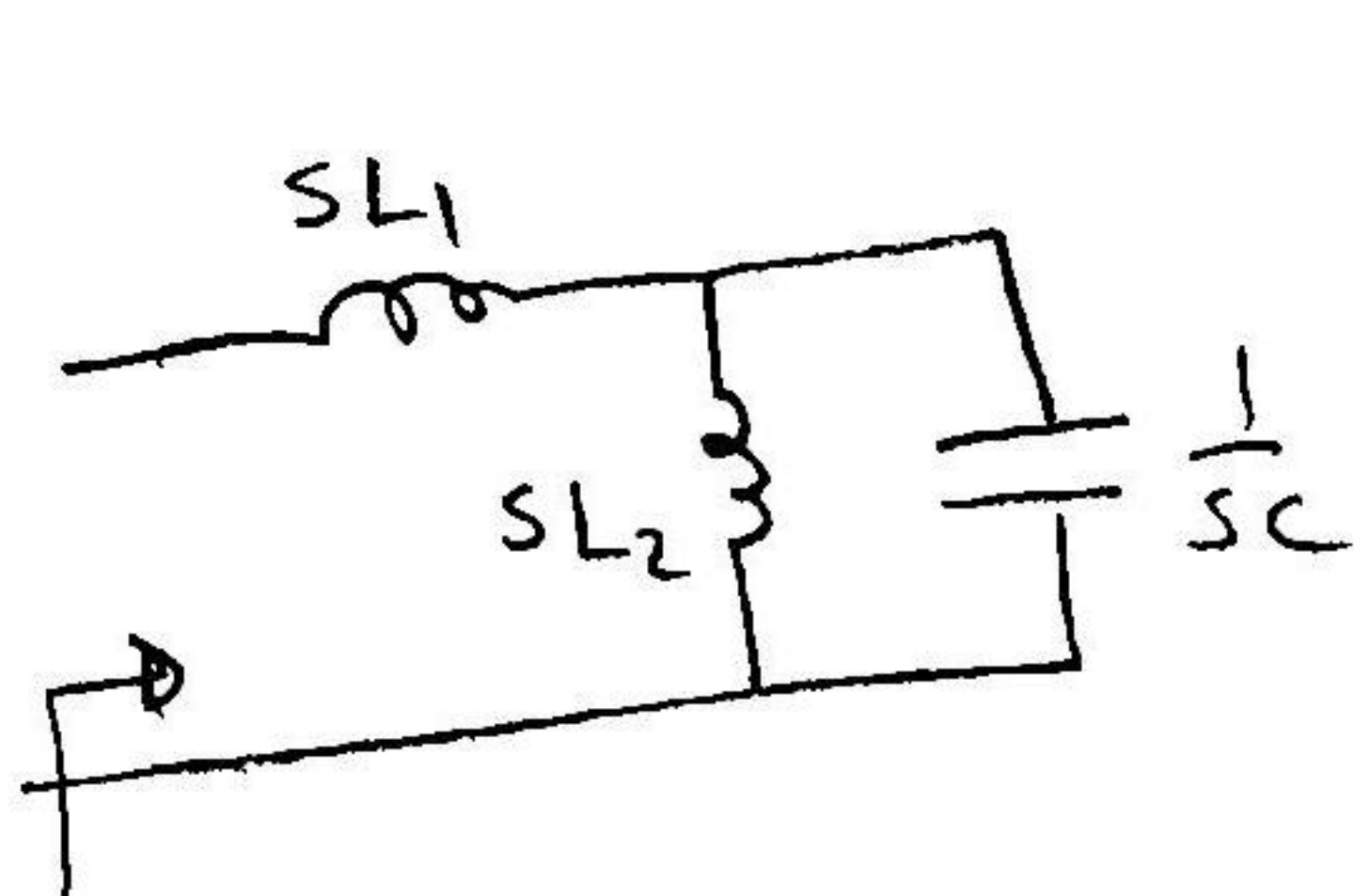
$$Z = \frac{R \cdot sL}{R + sL}$$



$$Z = \frac{sL \cdot \frac{1}{sC}}{sL + \frac{1}{sC}} = \frac{sL}{s^2 LC + 1}$$



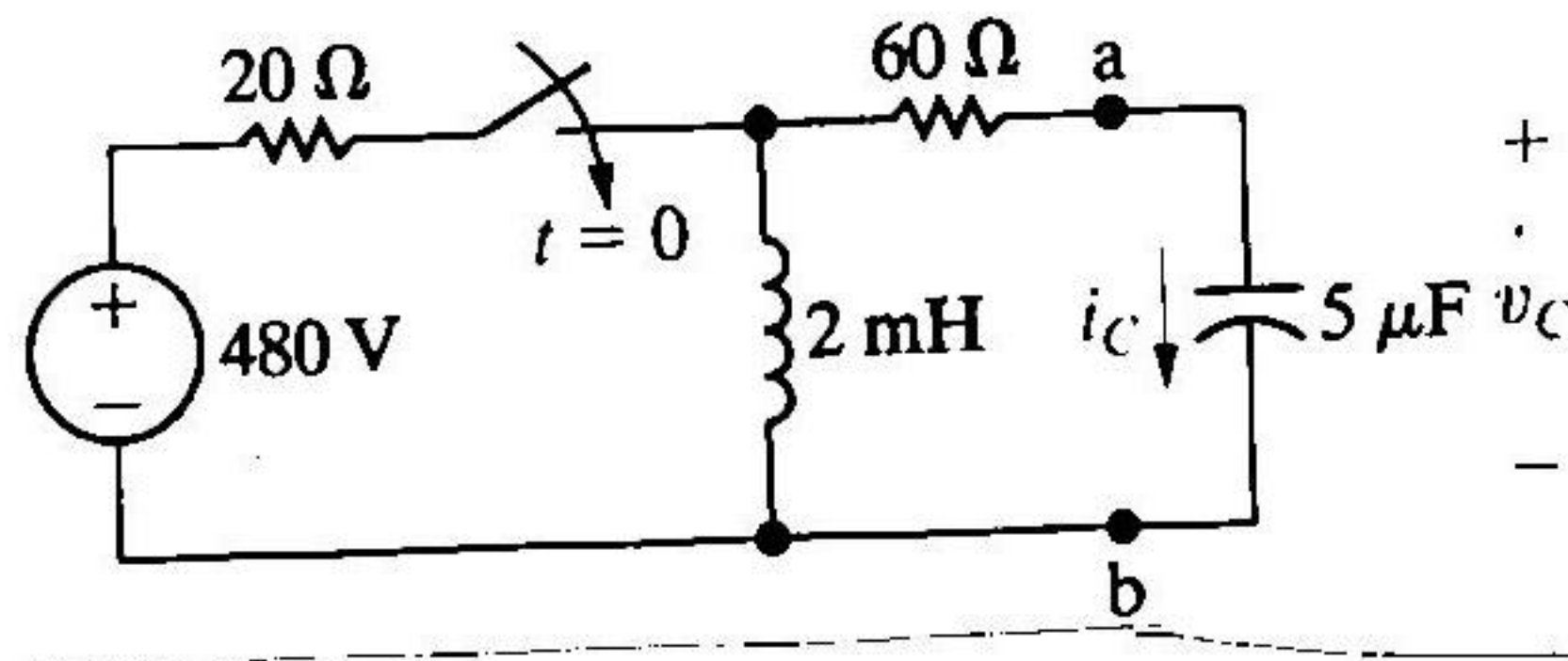
$$Z = sL + \frac{1}{sC} + R = \frac{s^2 LC + 1 + sCR}{sC}$$



$$Z = sL_1 + \frac{sL_2 \cdot \frac{1}{sC}}{sL_2 + \frac{1}{sC}} = sL_1 + \frac{sL_2}{s^2 L_2 C + 1}$$

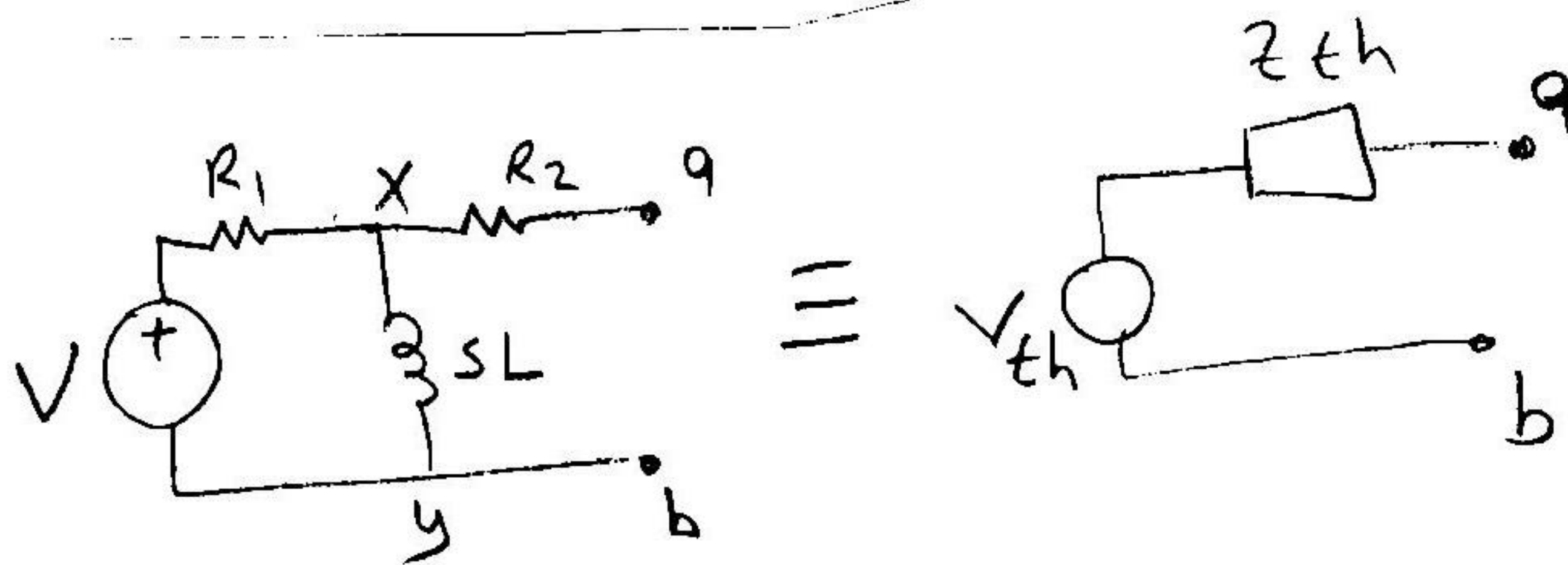
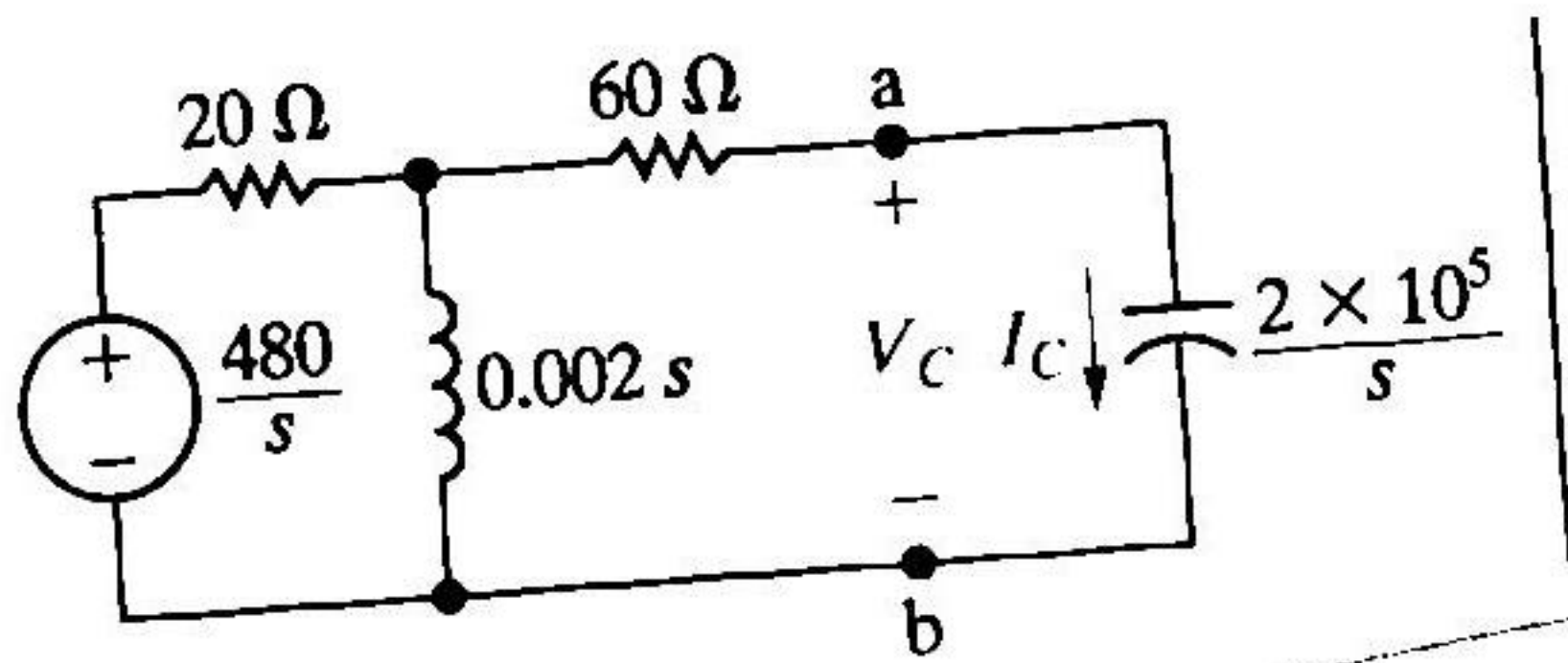
$$= \frac{s^3 L_1 L_2 C + sL_1 + sL_2}{s^2 L_2 C + 1}$$

Calculate Thévenin equivalent of the following circuit.

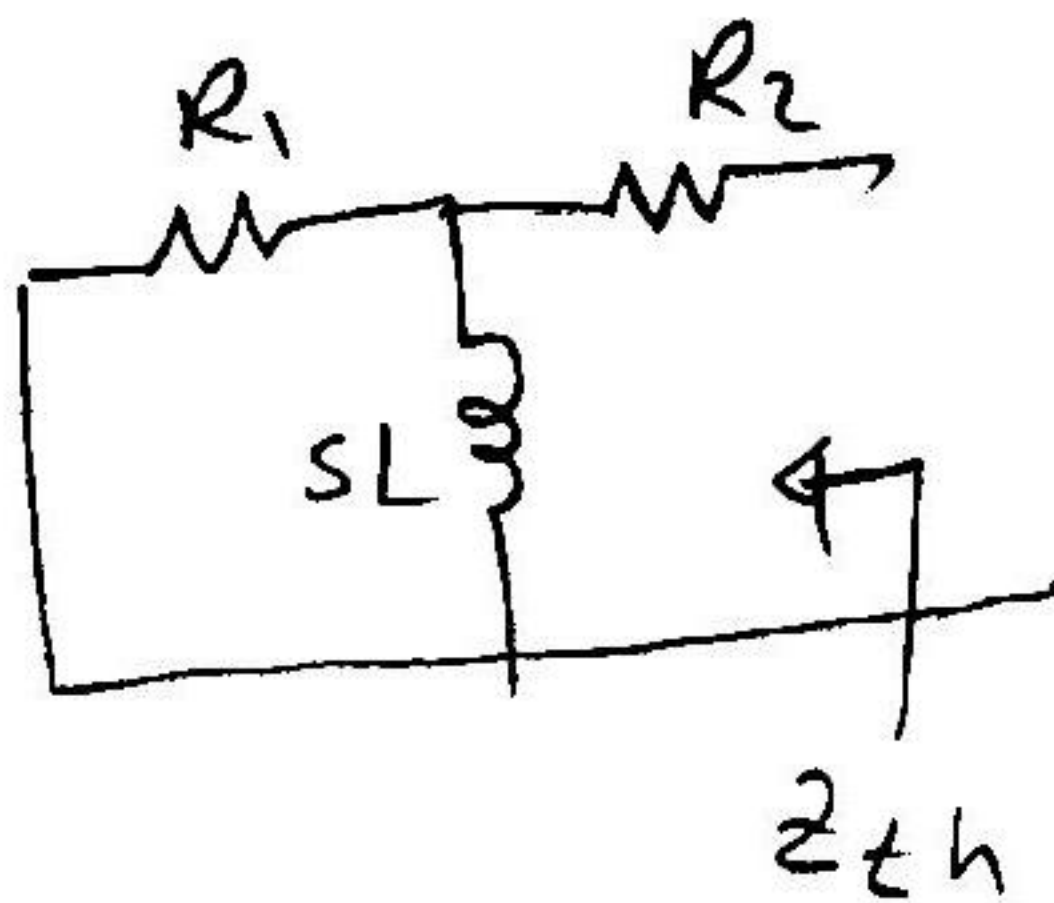


$$I_L(0) = 0 \quad V_c(0) = 0$$

Solution

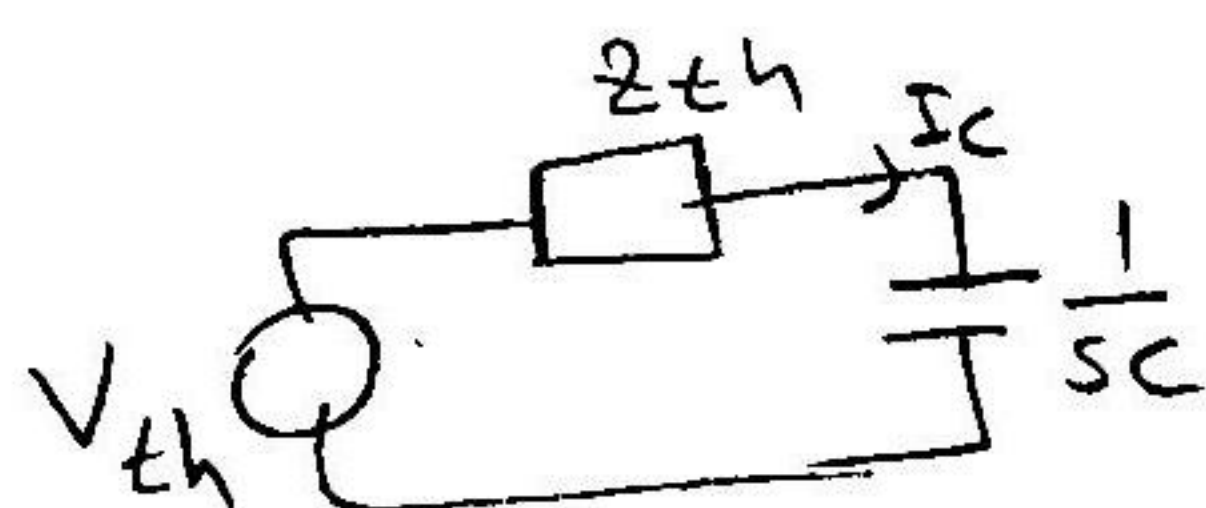


$$V_{th} = V_{ab} = V_{xy} = V \frac{sL}{sL + R_1} = \frac{480}{s} \frac{s \times 0.002}{s \times 0.002 + 20} = \frac{480}{s + 10000}$$



$$\begin{aligned} Z_{th} &= R_2 + \frac{sL R_1}{sL + R_1} = \frac{R_2 sL + R_2 R_1 + sL R_1}{sL + R_1} \\ &= \frac{60 \times s \times 0.002 + 60 \times 20 + s \times 0.002 \times 20}{s \times 0.002 + 20} \\ &= \frac{80(s + 7500)}{s + 10000} \end{aligned}$$

Calculate I_c if capacitor is connected



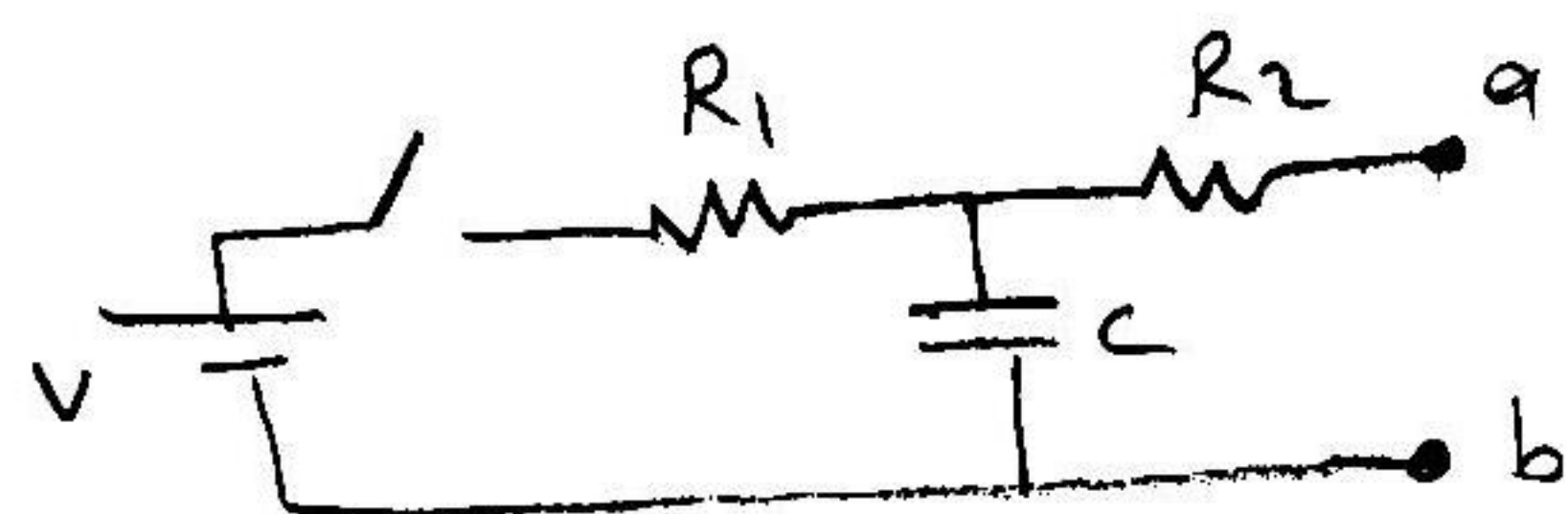
$$I_c = \frac{V_{th}}{Z_{th} + \frac{1}{sC}} = \frac{\frac{4800}{s + 10000}}{\frac{80(s + 7500)}{s + 10000} + \frac{1}{s \times 5 \times 10^{-6}}}$$

$$\begin{aligned} I_c(s) &= \frac{6}{s + 5000} - \frac{30000}{(s + 5000)^2} \rightarrow i_c(t) = 6e^{-5000t} - 30000te^{-5000t} \end{aligned}$$

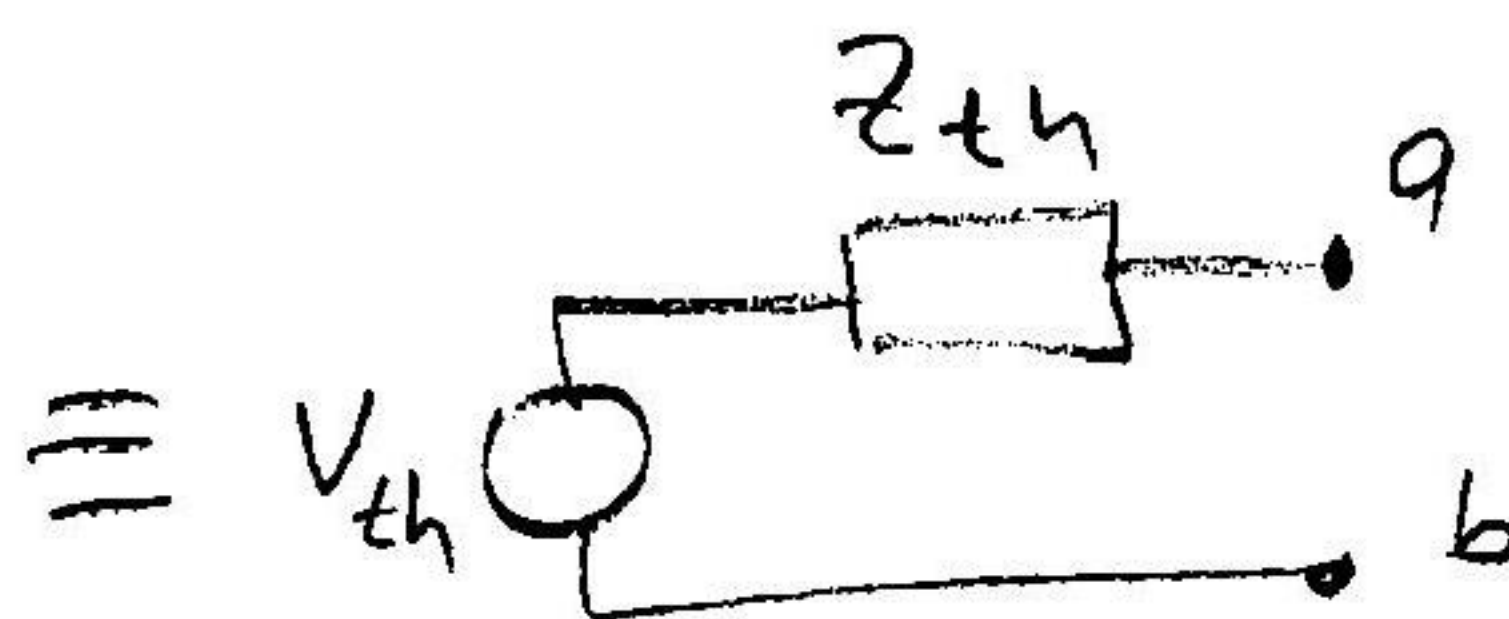
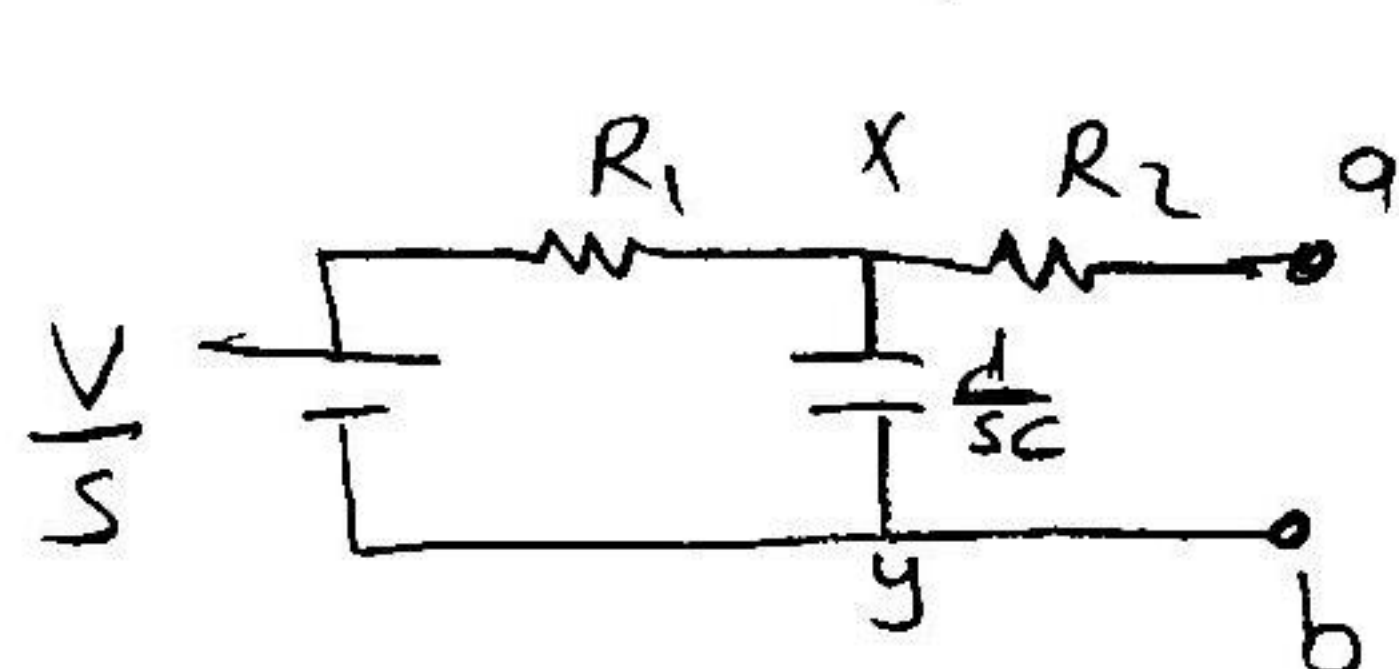
calculate Thevenin equivalent circuit.

386

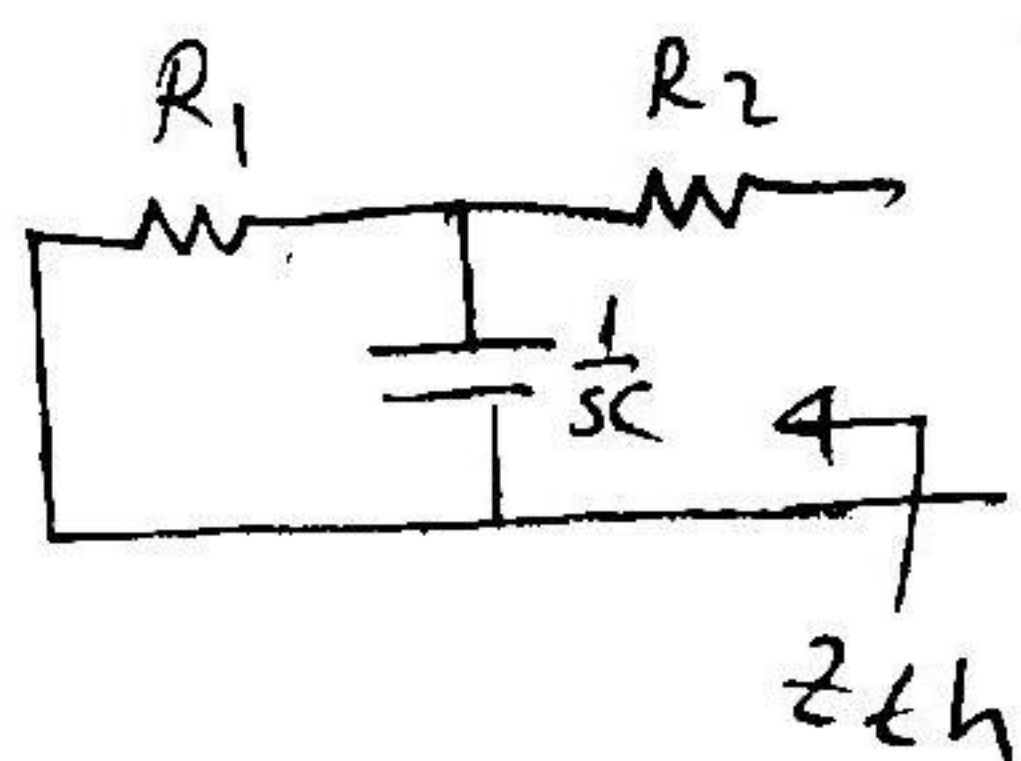
$$V_c(0) = 0$$



solution



$$V_{ab} = V_{th} = V_{xy} = \frac{V}{s} \frac{\frac{1}{sC}}{\frac{1}{sC} + R_1} = \frac{V}{s} \frac{1}{sCR + 1}$$



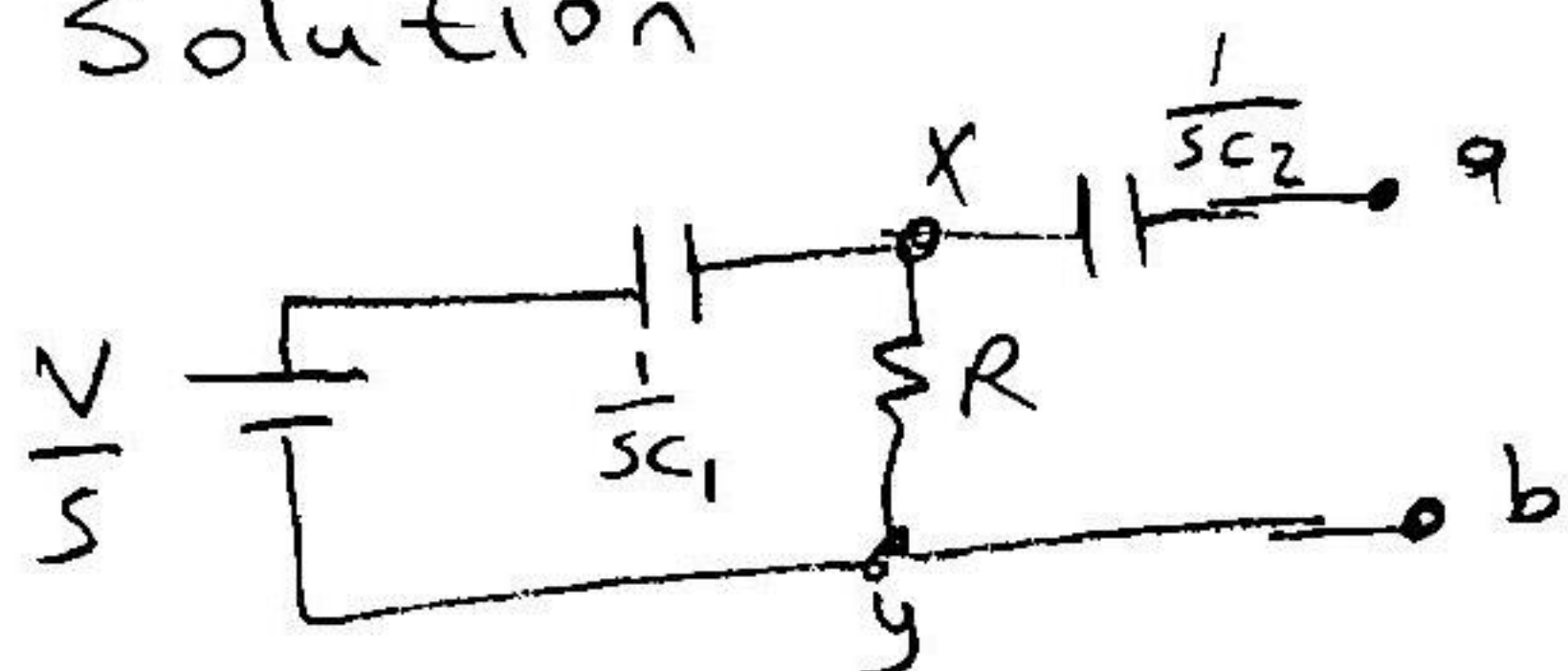
$$Z_{th} = R_2 + \frac{R_1 \frac{1}{sC}}{R_1 + \frac{1}{sC}} = R_2 + \frac{R_1}{sCR + 1}$$

$$Z_{th} = \frac{R_2 R_1 sC + R_2 + R_1}{sCR + 1}$$

calculate Thevenin equivalent circuit



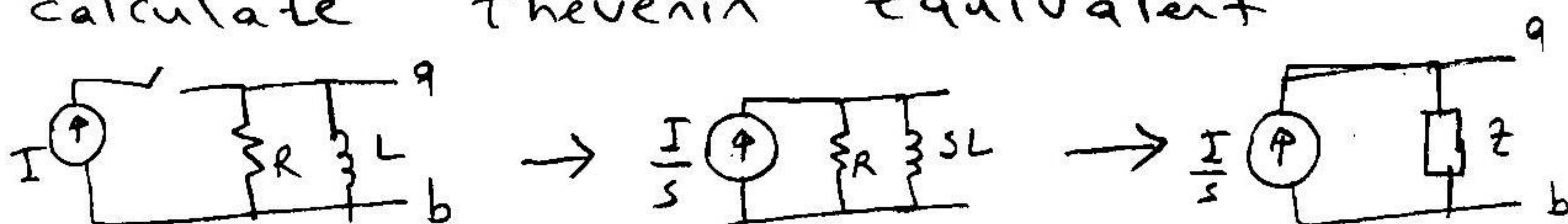
Solution



$$V_{ab} = V_{xy} = \frac{V}{s} \frac{R}{R + \frac{1}{sC_1}} = \frac{V}{s} \frac{sC_1 R}{sC_1 R + 1} = \frac{RC_1 V}{sC_1 R + 1}$$

$$Z_{th} = \frac{1}{sC_2} + \frac{R \cdot \frac{1}{sC_1}}{R + \frac{1}{sC_1}} = \frac{1}{sC_2} + \frac{R}{sC_1 R + 1}$$

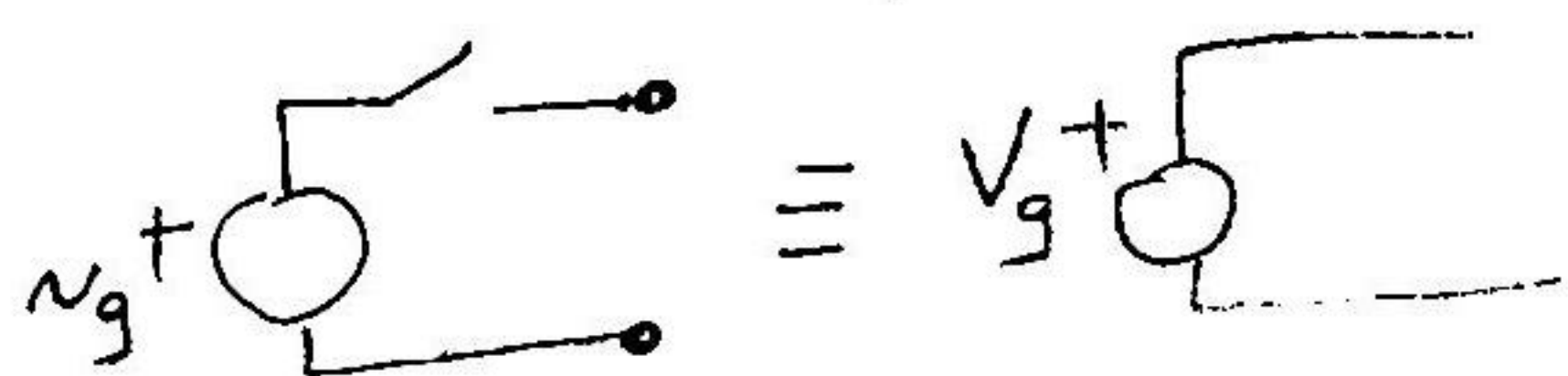
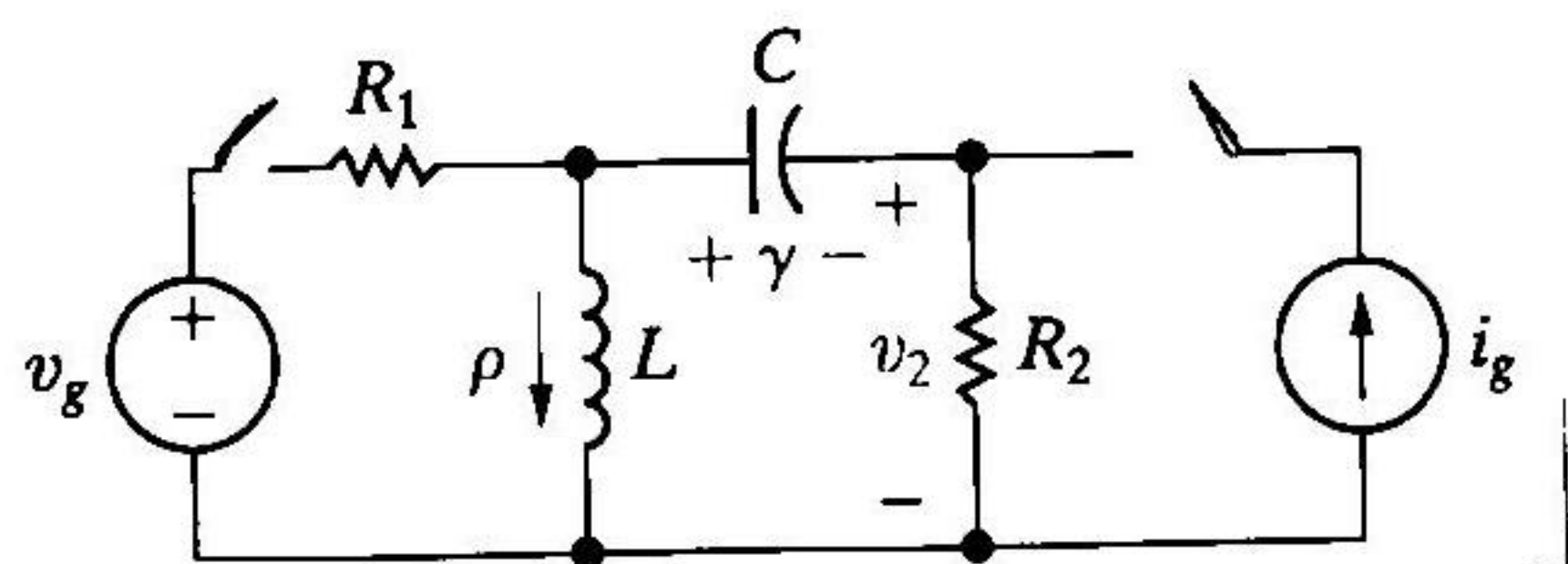
calculate Thevenin equivalent



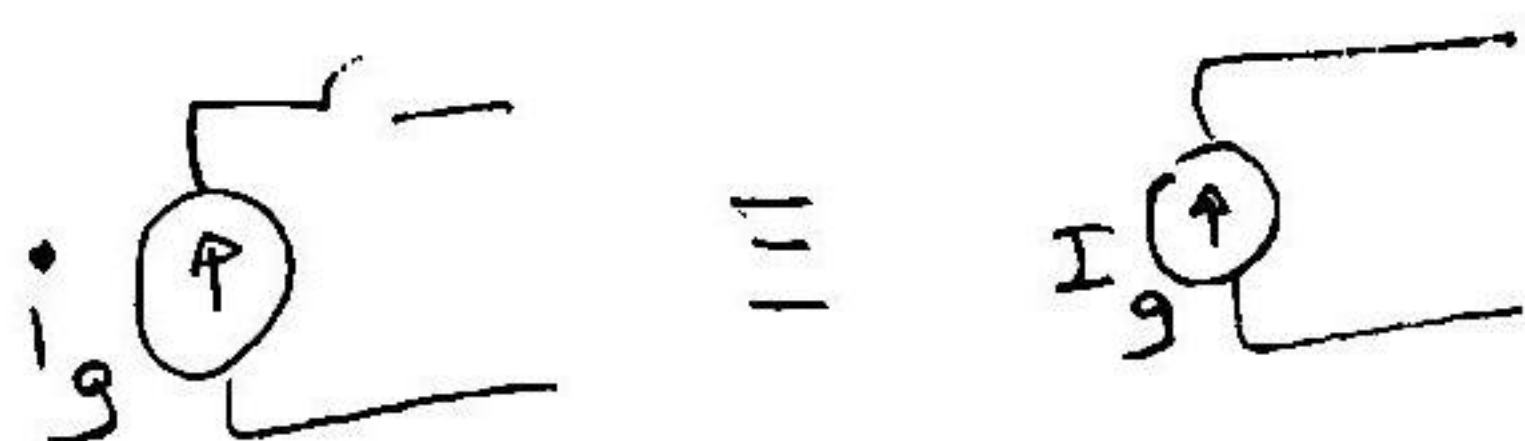
$$Z = \frac{sL R}{sL + R}$$

$$V_{th} = \frac{I}{s} \times Z = \frac{I}{s} \frac{sL R}{sL + R}$$

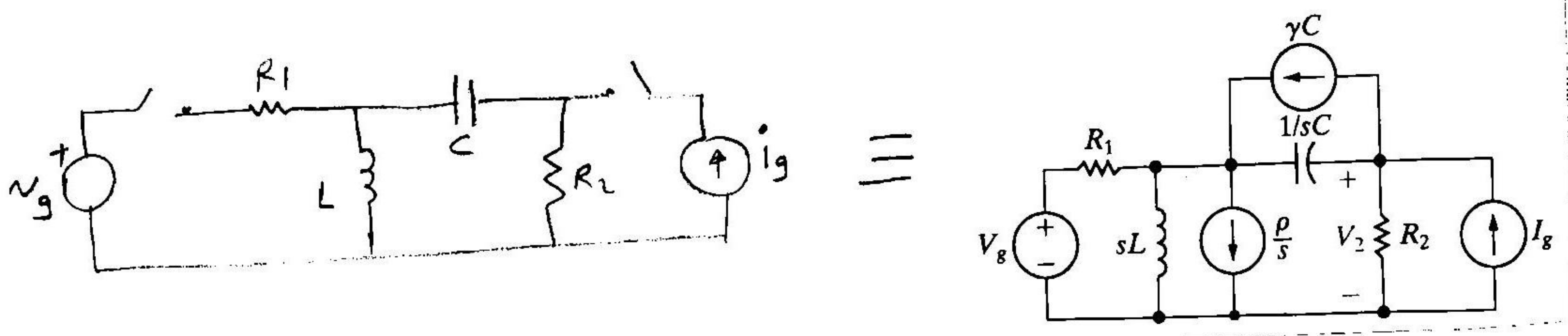
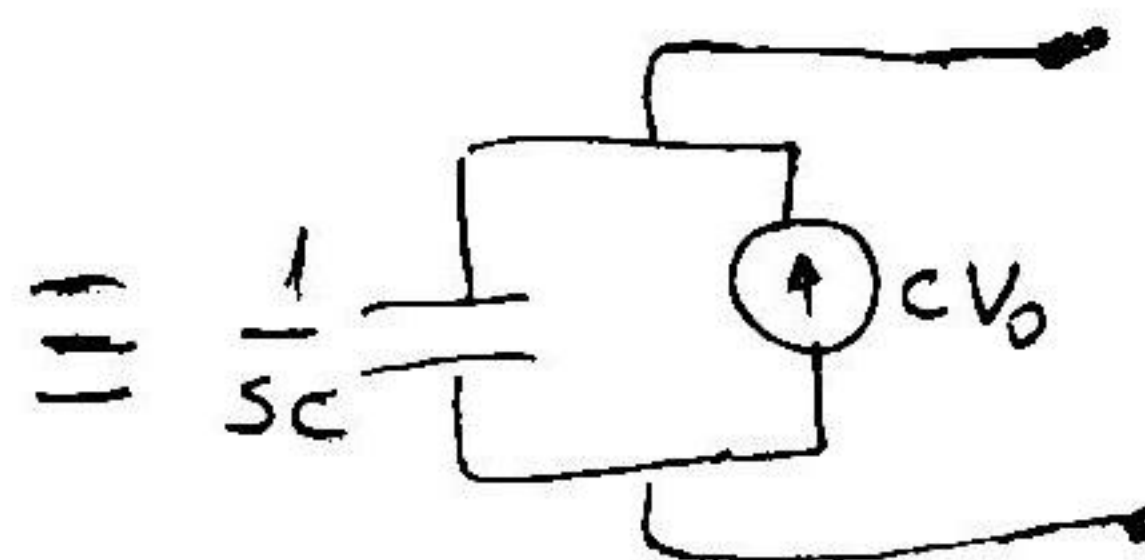
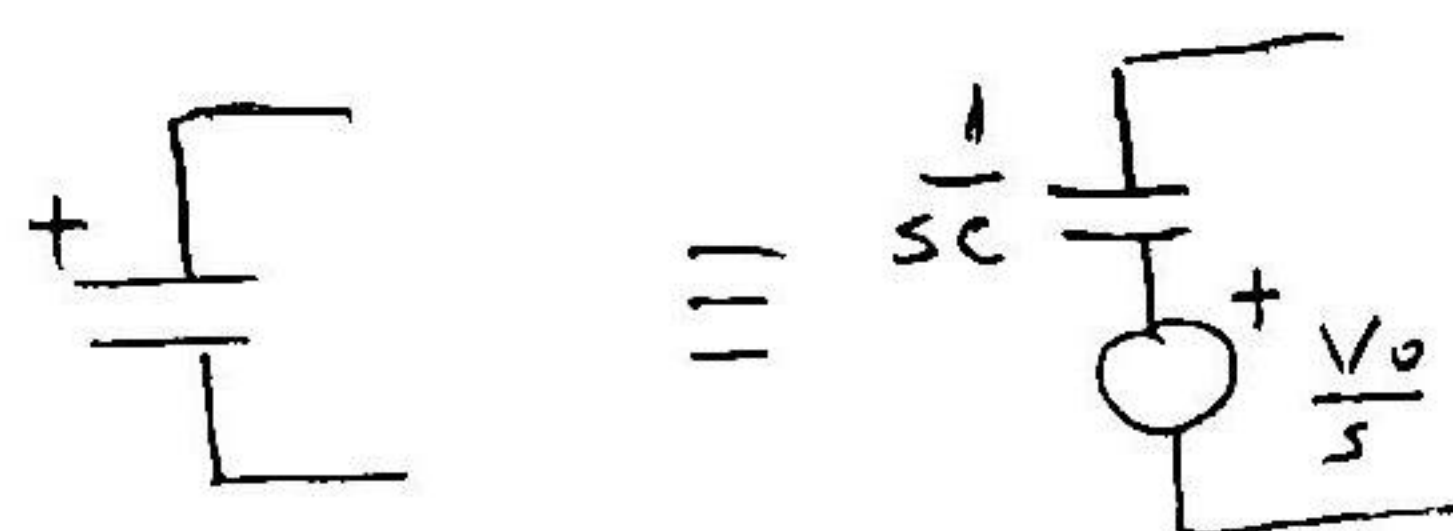
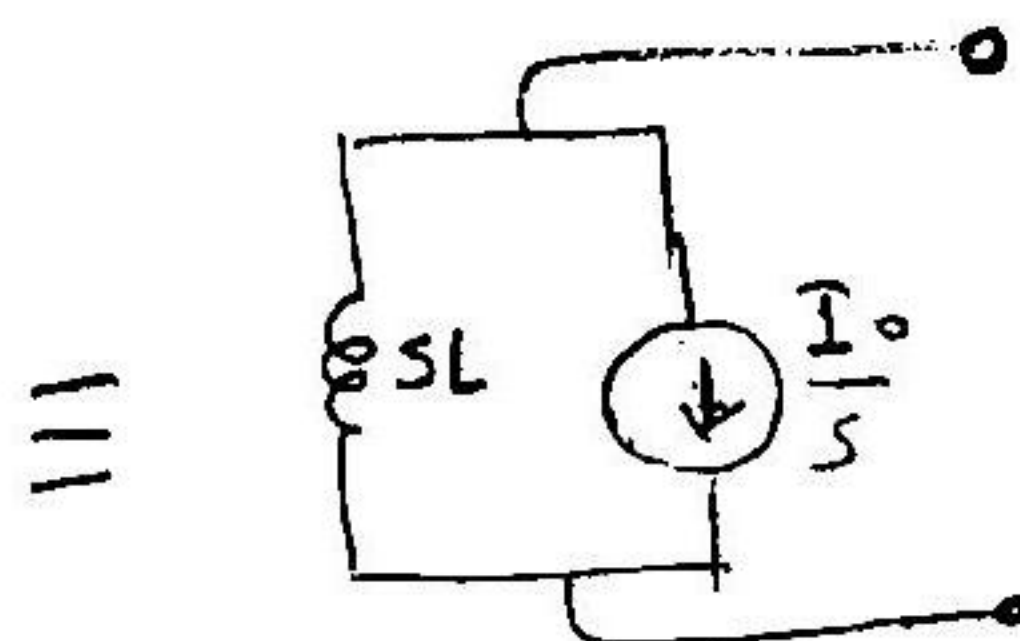
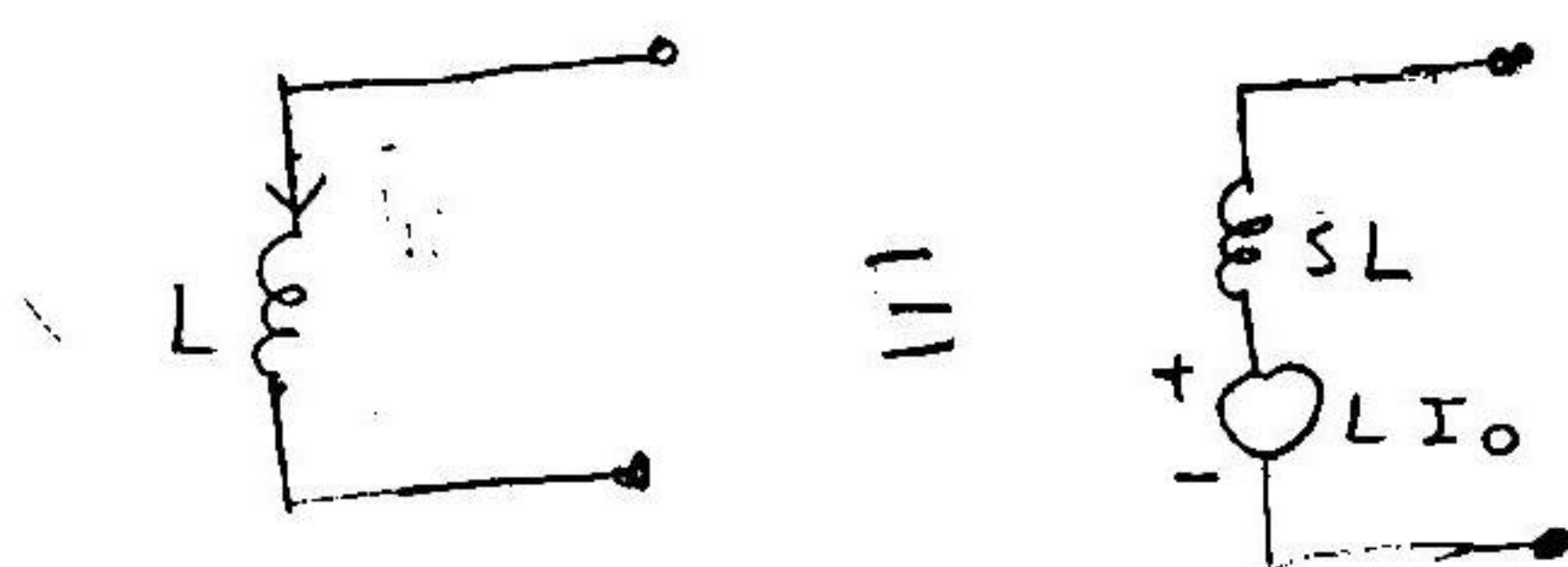
$$Z_{th} = Z = \frac{sL R}{sL + R}$$



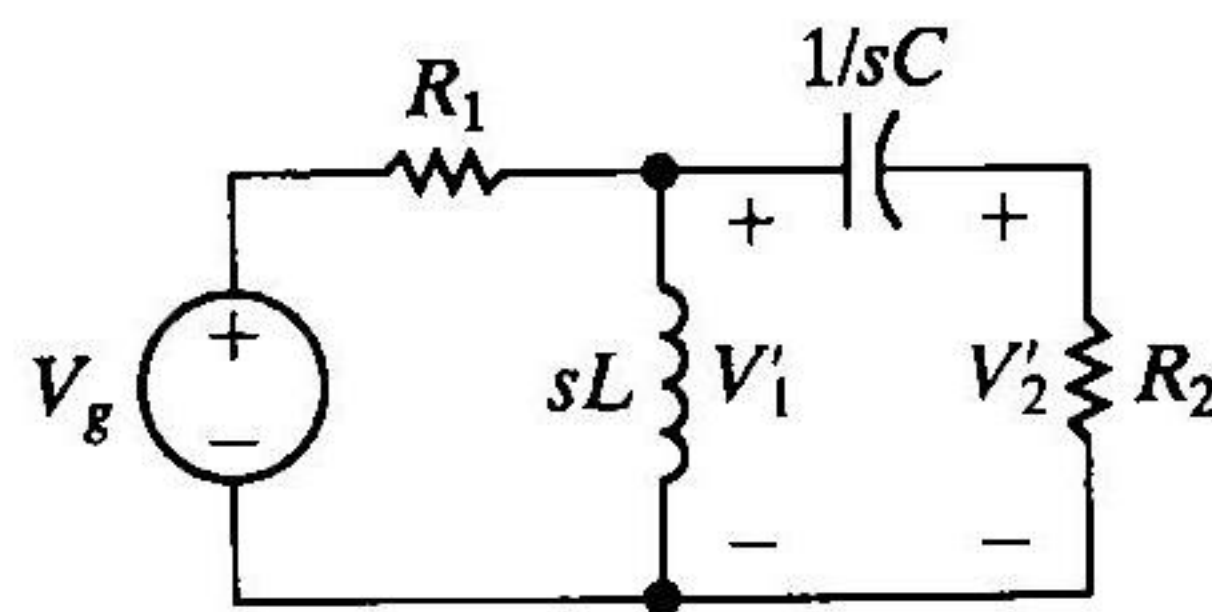
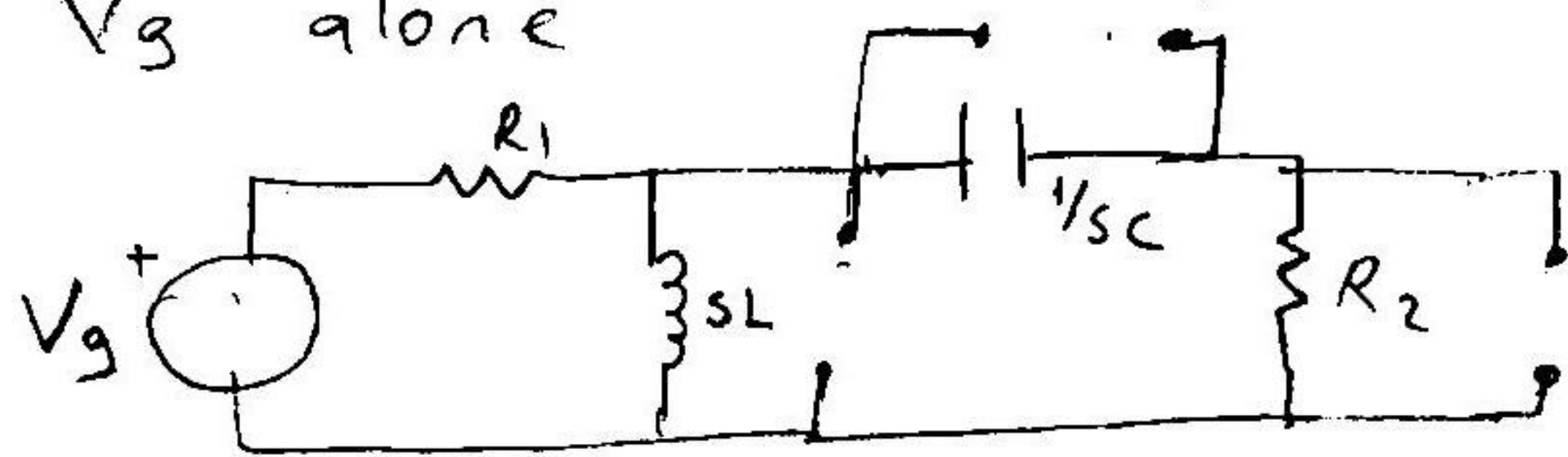
if $v_g = A u(t) \rightarrow V_g = \frac{A}{s}$



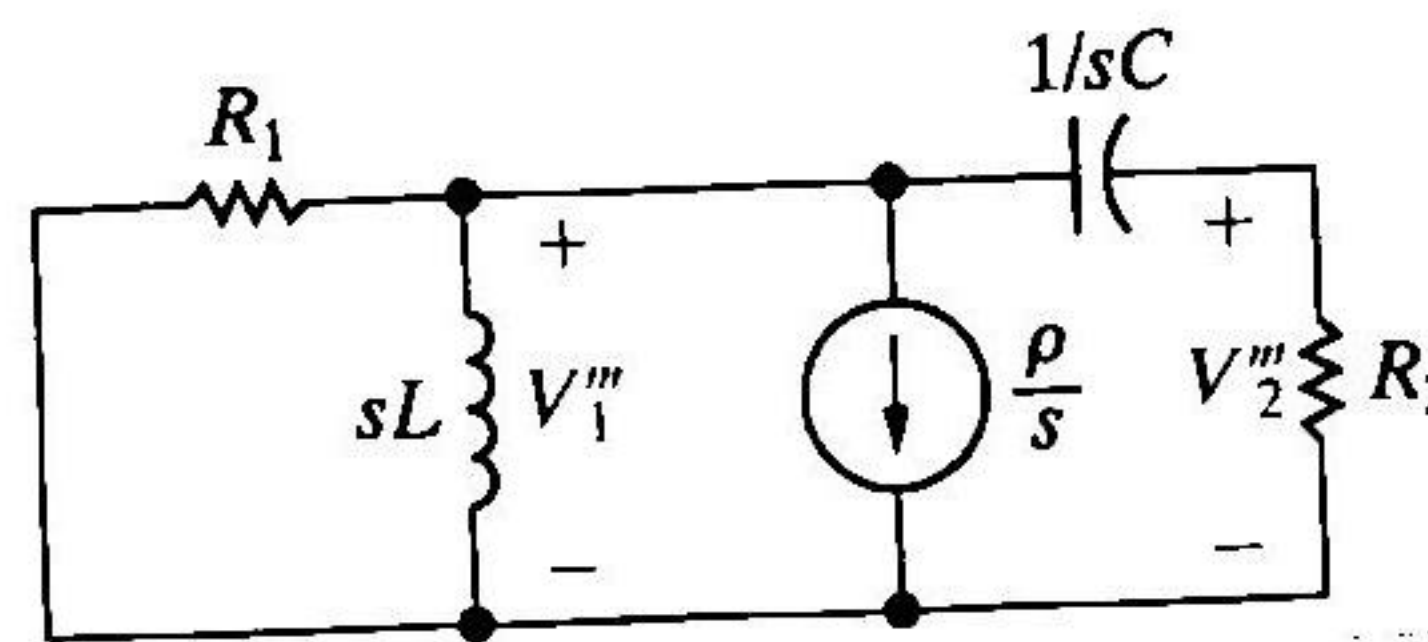
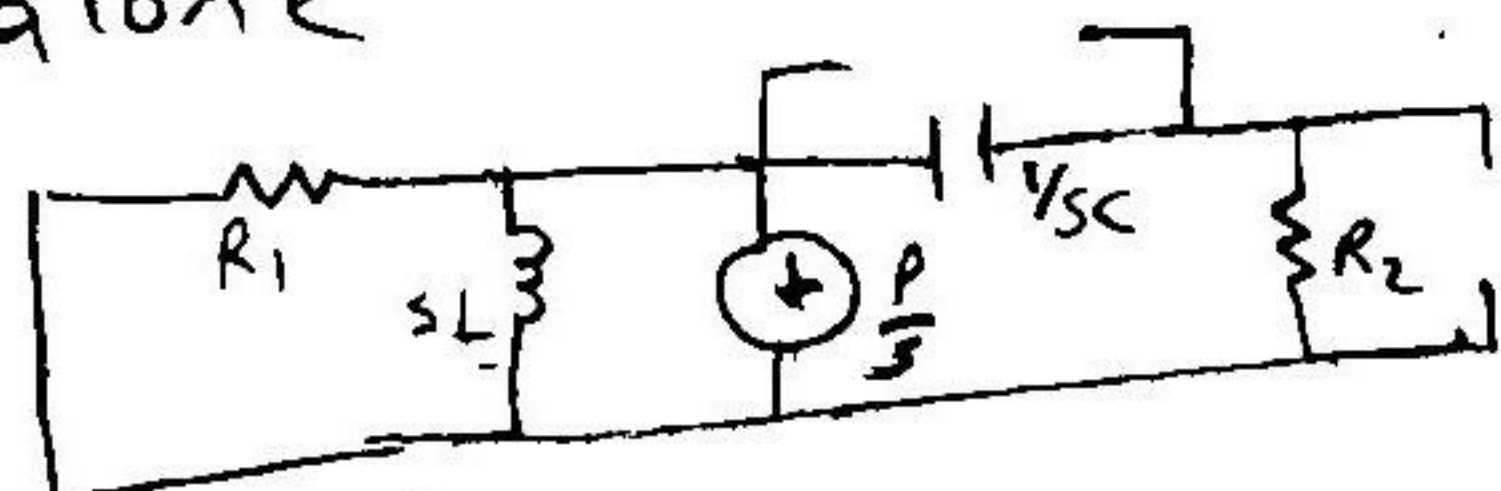
if $i_g = B u(t) \rightarrow I_g = \frac{B}{s}$



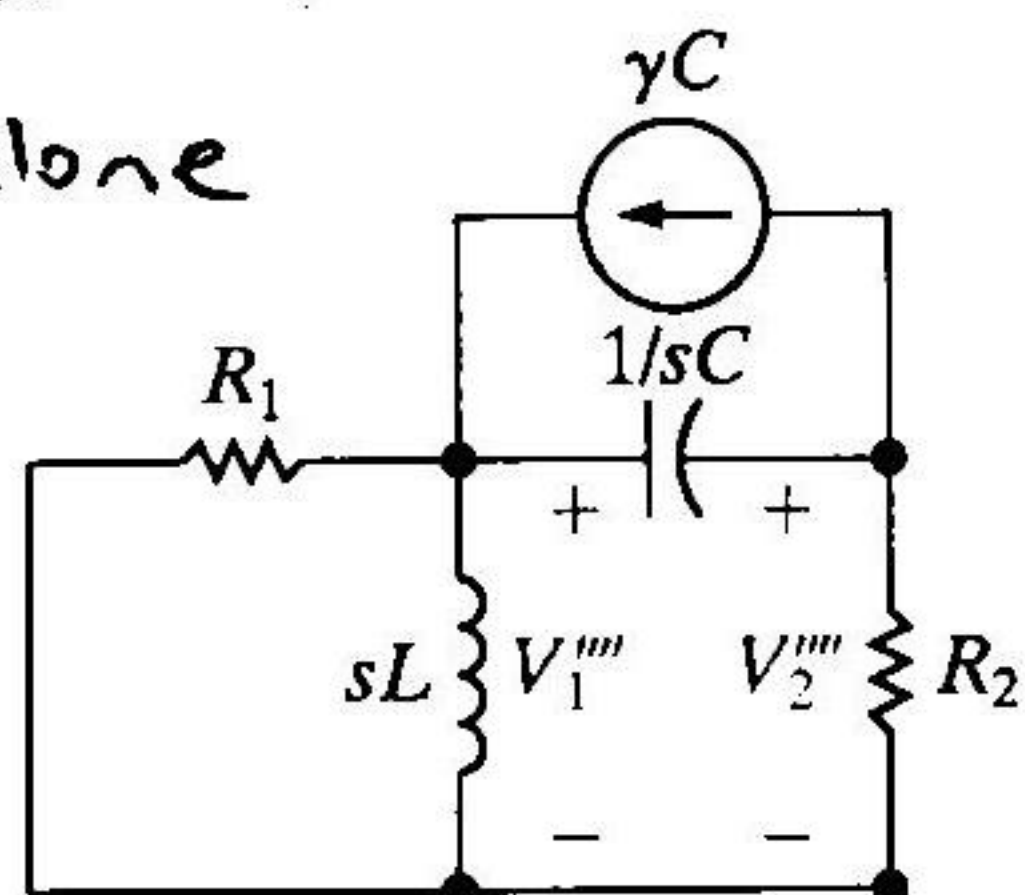
1) V_g alone



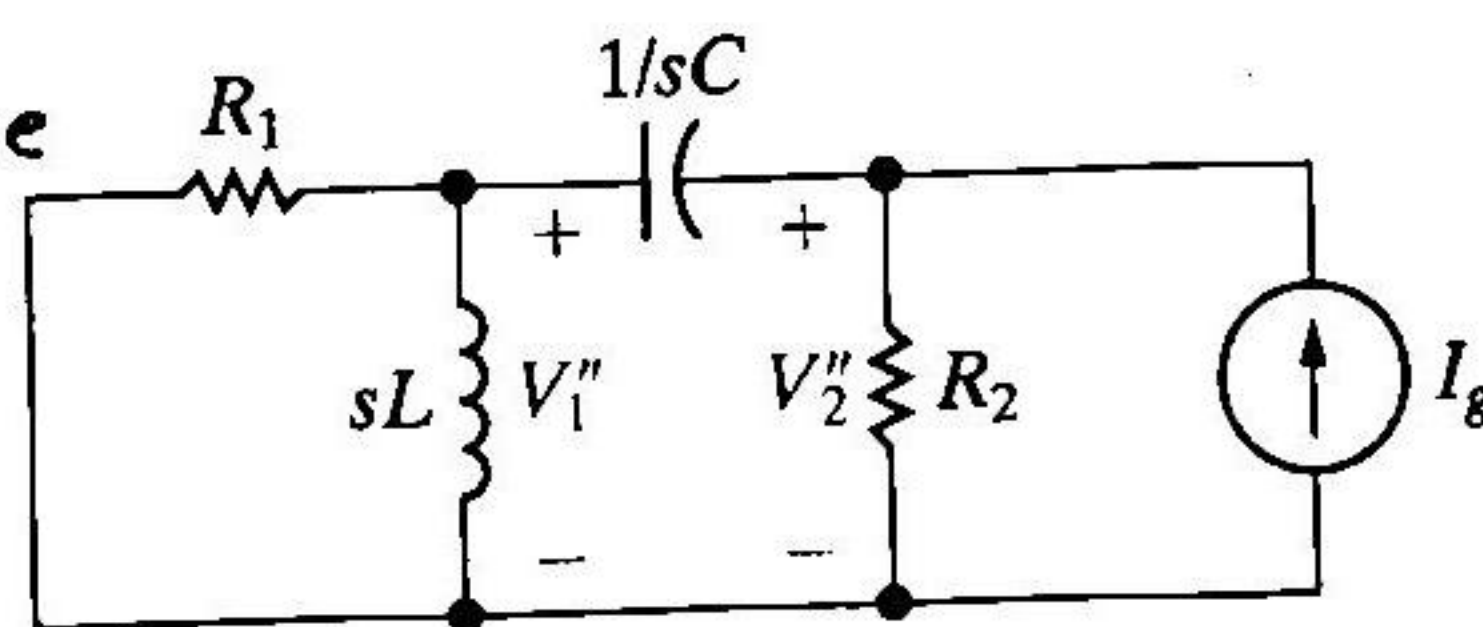
2) $\frac{\rho}{s}$ alone



3) γC alone



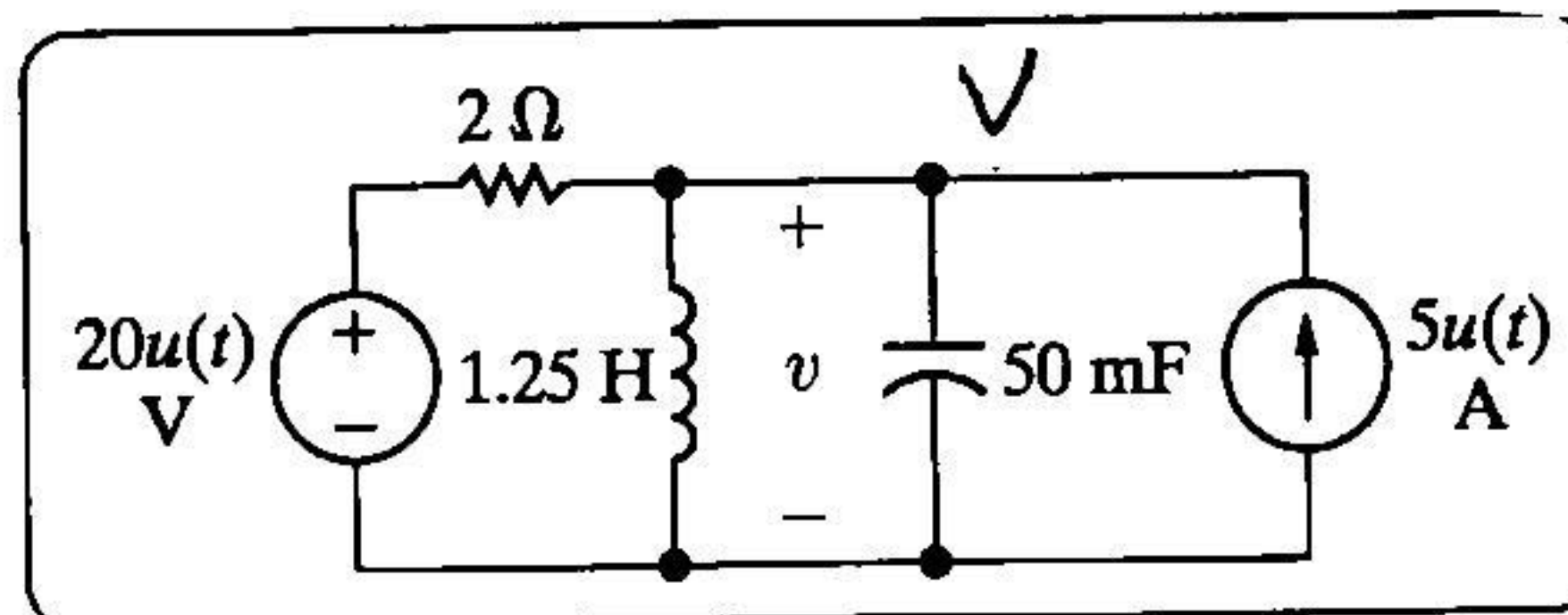
4) I_g alone



13.8

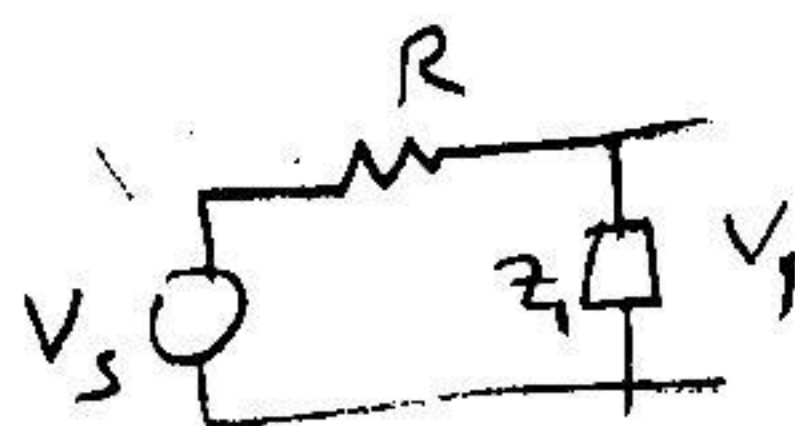
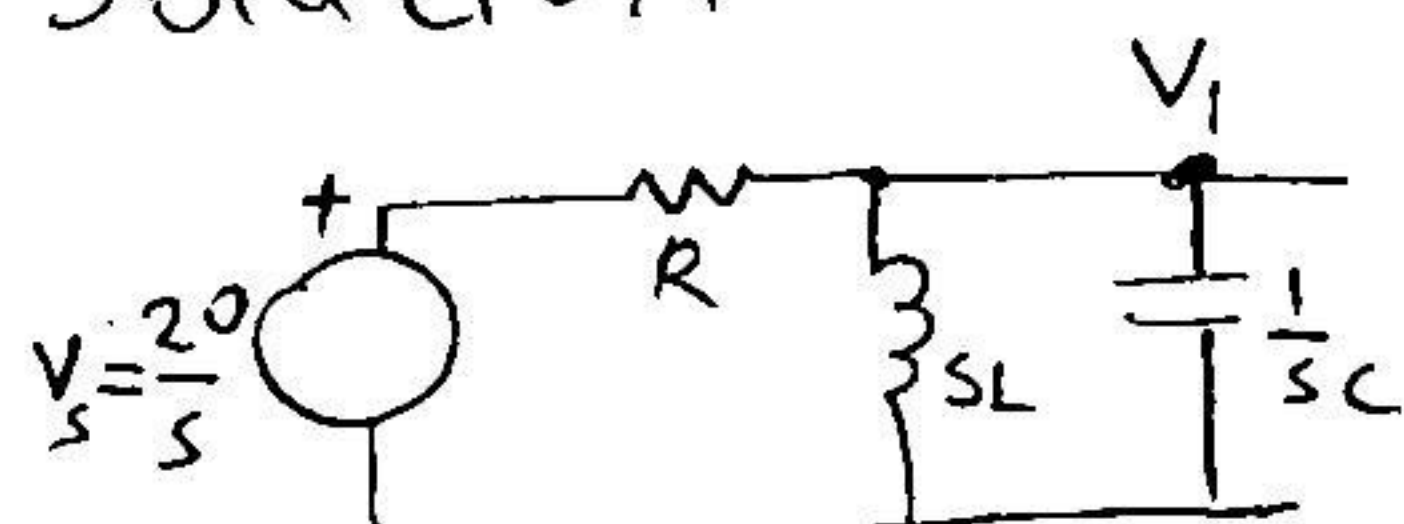
The energy stored in the circuit shown is zero at the instant the two sources are turned on.

- Find the component of v for $t > 0$ owing to the voltage source.
- Find the component of v for $t > 0$ owing to the current source.
- Find the expression for v when $t > 0$.



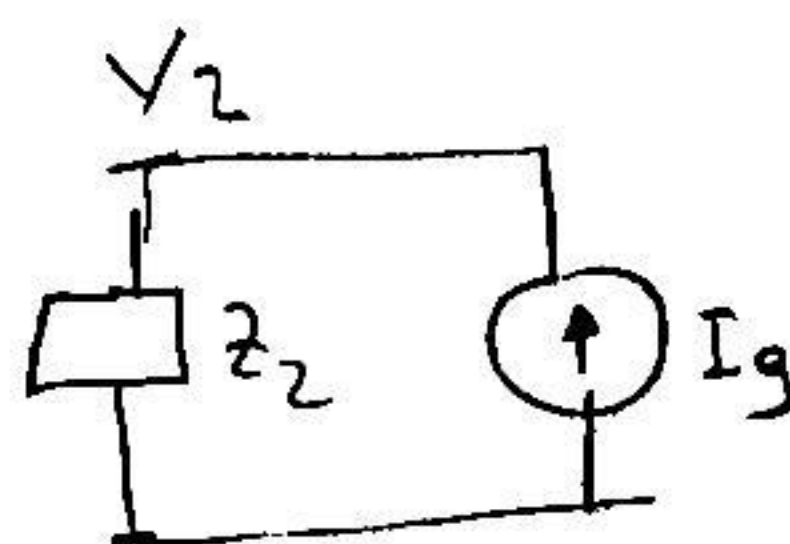
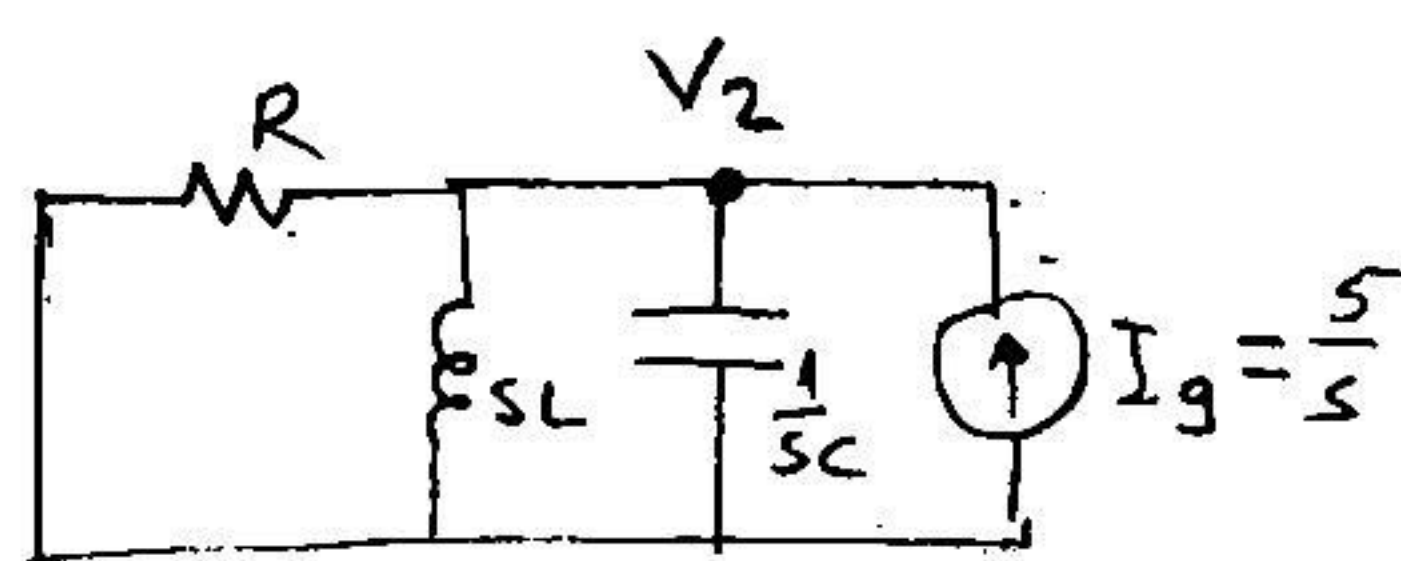
ANSWER: (a) $[(100/3)e^{-2t} - (100/3)e^{-8t}]u(t)$ V;
 (b) $[(50/3)e^{-2t} - (50/3)e^{-8t}]u(t)$ V;
 (c) $[50e^{-2t} - 50e^{-8t}]u(t)$ V.

Solution



$$V_1 = V_s \frac{Z_1}{Z_1 + R}$$

$$Z_1 = \frac{sL \frac{1}{sC}}{sL + \frac{1}{sC}} = \frac{sL}{s^2LC + 1}$$



$$V_2 = Z_2 I_g$$

$$\frac{1}{Z_2} = \frac{1}{R} + \frac{1}{sL} + \frac{1}{\frac{1}{sC}}$$

$$V = V_1 + V_2$$

calculations

$$Z_1 = \frac{sL}{s^2LC + 1} = \frac{1.25s}{s^2 \cdot 1.25 \cdot 50 \cdot 10^{-3} + 1} = \frac{1.25s}{0.0625s^2 + 1} = \frac{20s}{s^2 + 16}$$

$$V_1 = V_s \frac{Z_1}{Z_1 + R} = \frac{20}{s} \frac{\frac{20s}{s^2 + 16}}{\frac{20s}{s^2 + 16} + 2} = \frac{20}{s} \frac{20s}{20s + 2(s^2 + 16)} = \frac{400}{2s^2 + 20s + 32}$$

$$\frac{1}{Z_2} = \frac{1}{2} + \frac{1}{1.25s} + s \cdot 50 \cdot 10^{-3} = 0.5 + \frac{0.8}{s} + 0.05s = \frac{0.05s^2 + 0.5s + 0.8}{s}$$

$$Z_2 = \frac{s}{0.05s^2 + 0.5s + 0.8} = \frac{20s}{s^2 + 10s + 16}$$

$$V_2 = Z_2 I_g = \frac{20s}{s^2 + 10s + 16} \cdot \frac{5}{s} = \frac{100}{s^2 + 10s + 16}$$

$$V = V_1 + V_2 = \frac{400}{2(s^2 + 10s + 16)} + \frac{100}{s^2 + 10s + 16} = \frac{300}{s^2 + 10s + 16} = \frac{A}{(s+8)} + \frac{B}{s+2}$$

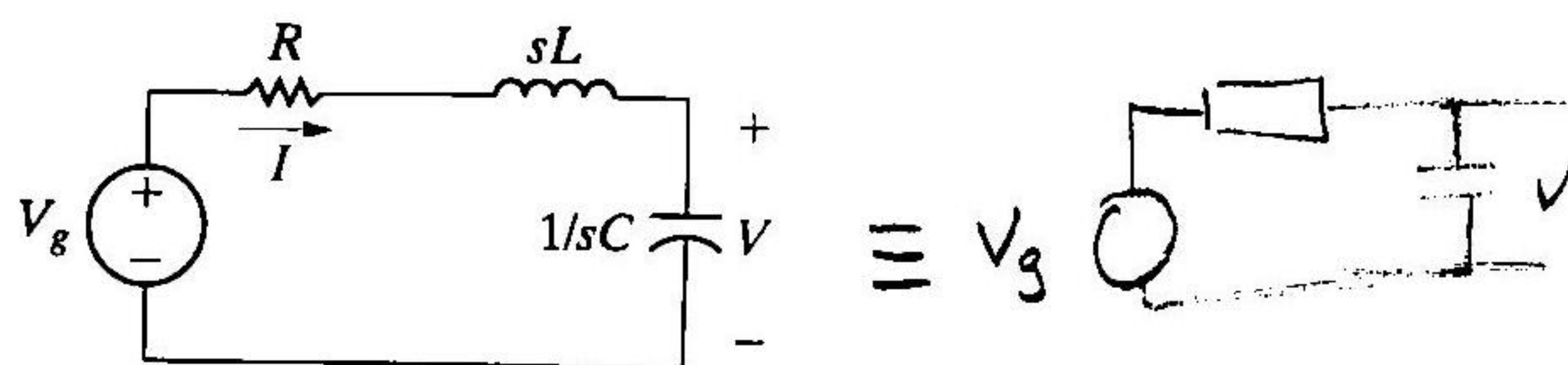
$$A = -50 \quad B = 50$$

$$v(t) = -50e^{-8t} + 50e^{-2t}$$

13.4 ♦ The Transfer Function

$$H(s) = \frac{Y(s)}{X(s)}, \quad (13.93)$$

$Y(s)$ is the Laplace transform of the output signal, and $X(s)$ is the Laplace transform of the input signal.



$$H(s) = \frac{V}{V_g} = \frac{1/sC}{R + sL + 1/sC} = \frac{1}{s^2LC + RCs + 1}$$

Figure 13.30 A series RLC circuit.

EXAMPLE 13.1

The voltage source v_g drives the circuit shown in Fig. 13.31. The response signal is the voltage across the capacitor, v_o .

- Calculate the numerical expression for the transfer function.
- Calculate the numerical values for the poles and zeros of the transfer function.

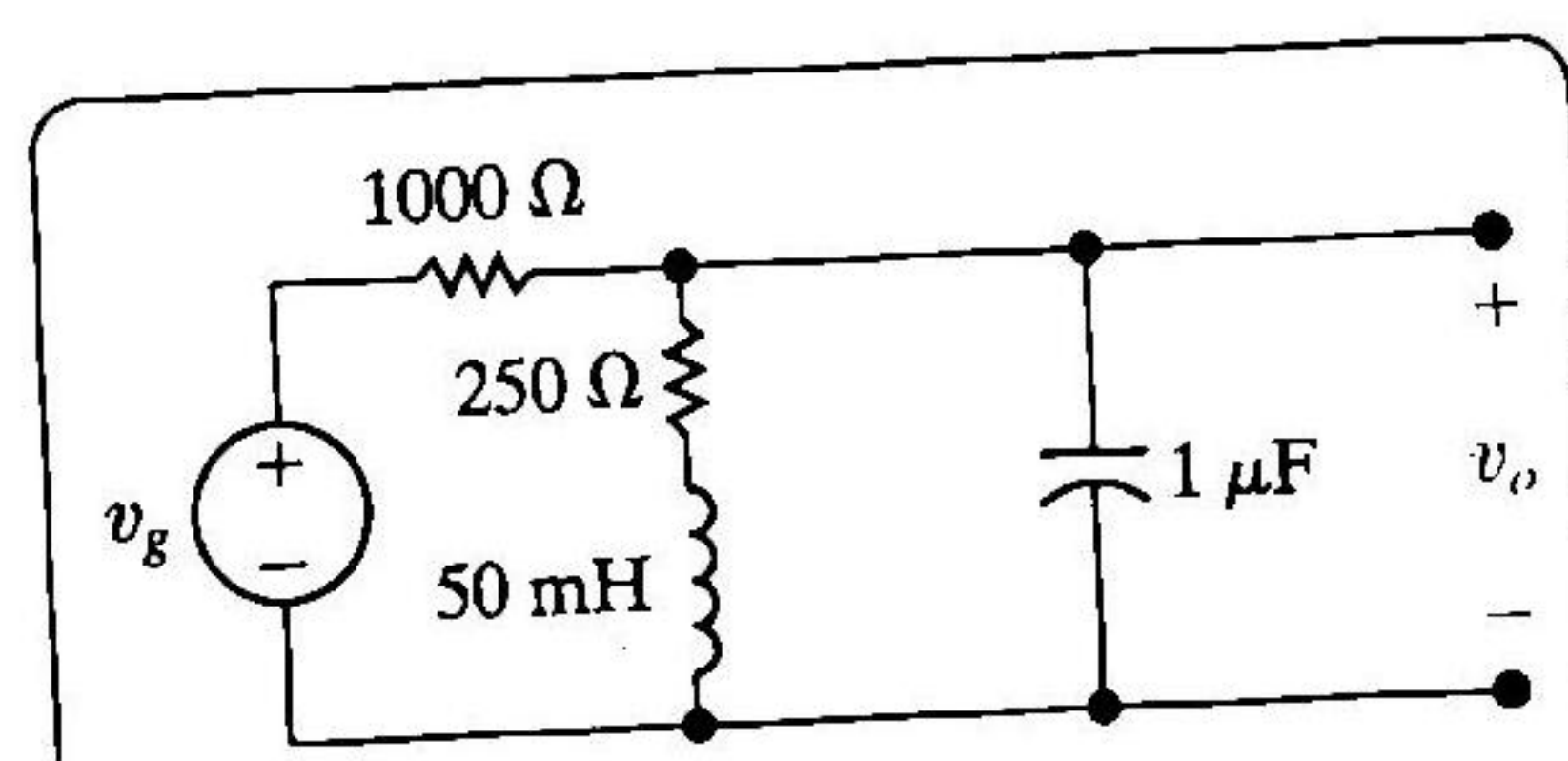
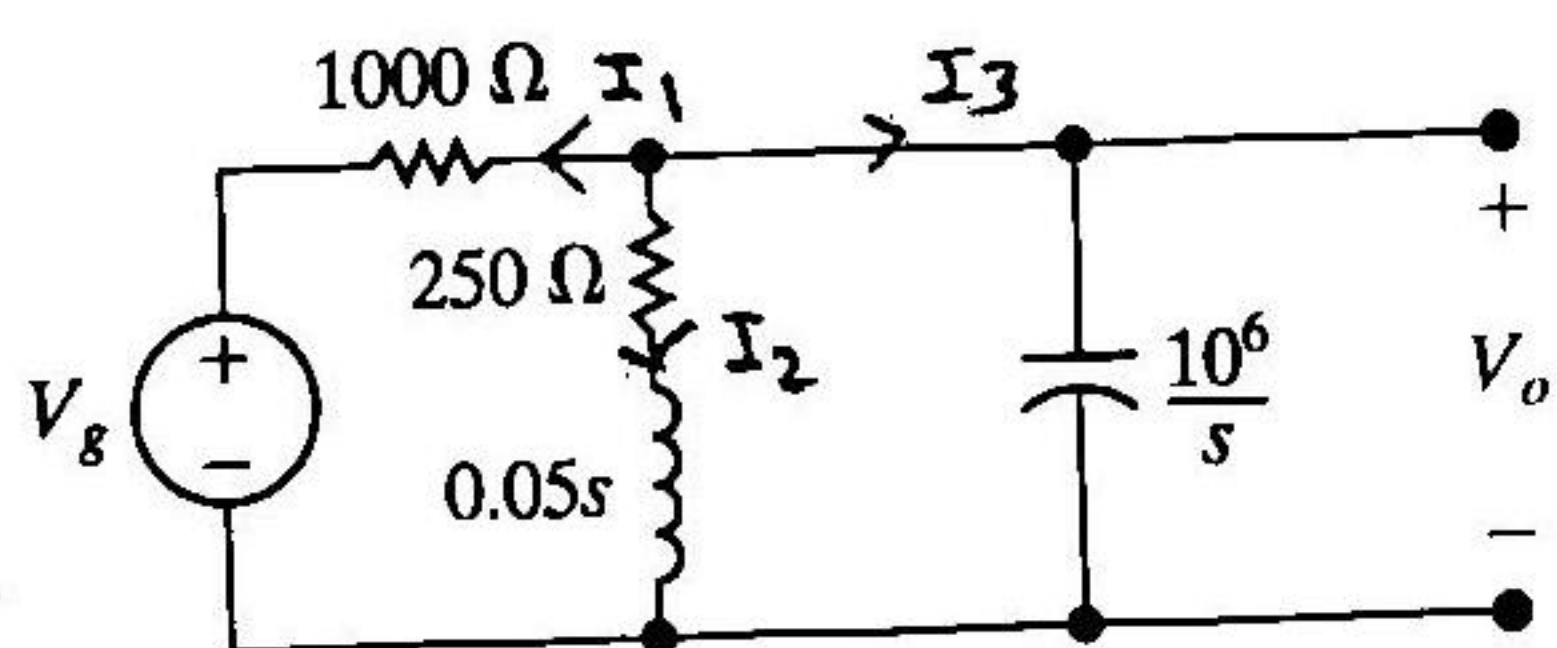


Figure 13.31 The circuit for Example 13.1.

SOLUTION



$$I_1 + I_2 + I_3 = 0$$

$$\frac{V_o - V_g}{1000} + \frac{V_o}{250 + 0.05s} + \frac{V_o s}{10^6} = 0$$

$$V_o \left(\frac{1}{1000} + \frac{1}{250 + 0.05s} + \frac{s}{10^6} \right) - \frac{V_g}{1000} = 0$$

$$V_o = \frac{V_g}{1000} \frac{1}{\frac{1}{1000} + \frac{1}{250 + 0.05s} + \frac{s}{10^6}}$$

$$V_o = \frac{1000(s + 5000)V_g}{s^2 + 6000s + 25 \times 10^6}$$

Hence the transfer function is

$$H(s) = \frac{V_o}{V_g} = \frac{1000(s + 5000)}{s^2 + 6000s + 25 \times 10^6}$$

- The poles of $H(s)$ are the roots of the denominator polynomial. Therefore

$$-p_1 = -3000 - j4000,$$

$$-p_2 = -3000 + j4000.$$

The zeros of $H(s)$ are the roots of the numerator polynomial; thus $H(s)$ has a zero at

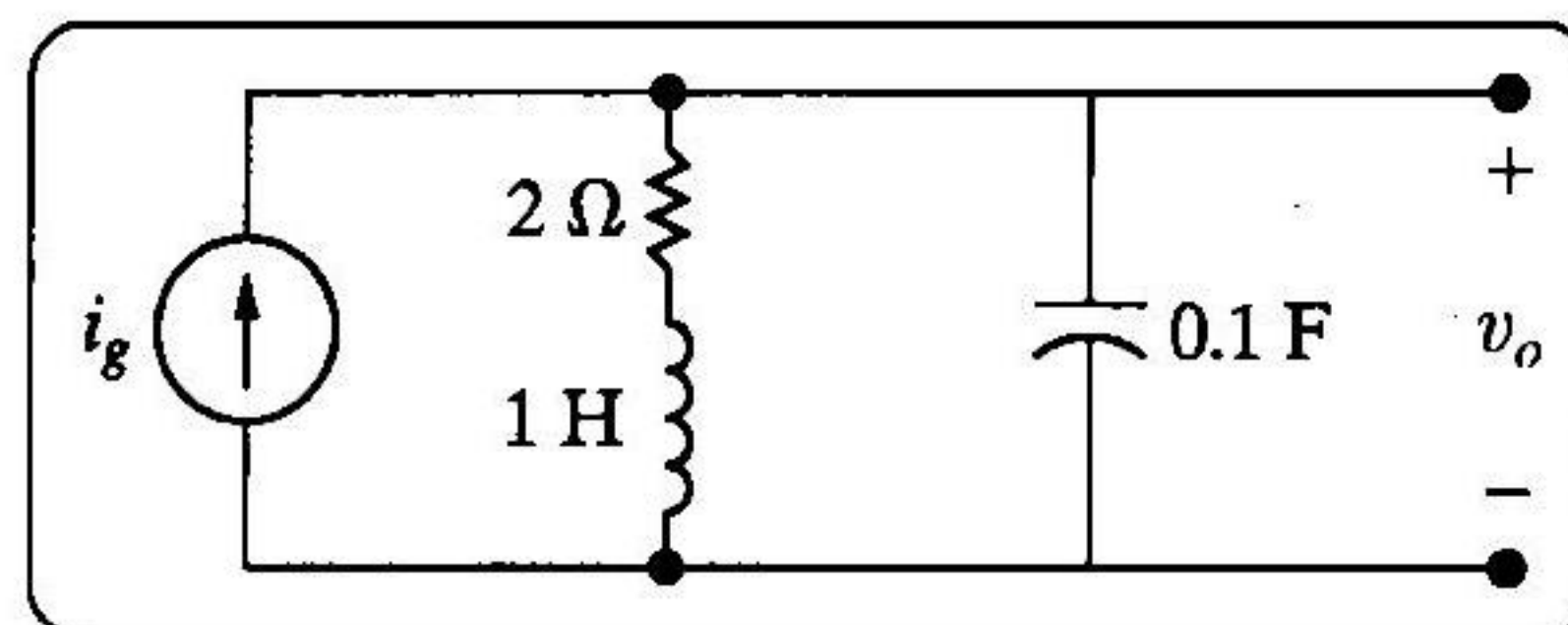
$$-z_1 = -5000.$$

Poles = Denominator roots

Zeros = Numerator roots

13.9

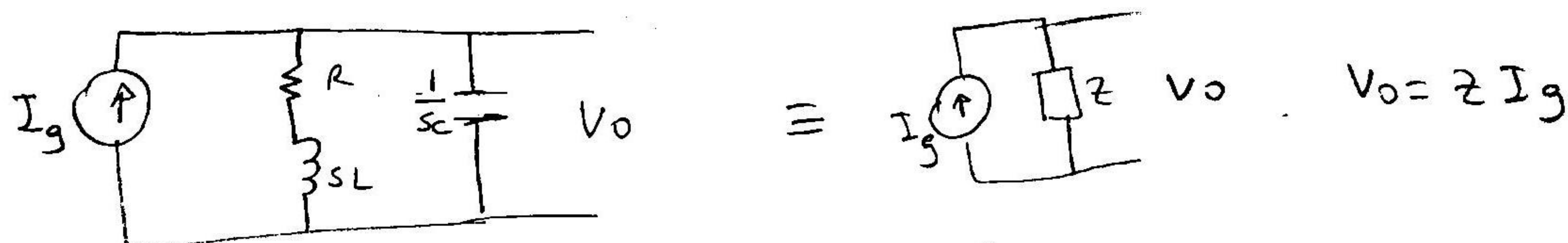
- a) Derive the numerical expression for the transfer function V_o/I_g for the circuit shown.
- b) Give the numerical value of each pole and zero of $H(s)$.



380

ANSWER: (a) $H(s) = 10(s+2)/(s^2+2s+10)$;
 (b) $-p_1 = -1+j3$, $-p_2 = -1-j3$,
 $-z = -2$.

Solution



$$V_o = Z I_g \rightarrow \frac{V_o}{I_g} = Z \quad H(s) = \frac{V_o}{I_g} = Z$$

$$Z = (R + sL) \parallel \frac{1}{sC} = \frac{(R + sL) \frac{1}{sC}}{(R + sL) + \frac{1}{sC}} = \frac{R + sL}{(R + sL)sC + 1}$$

$$= \frac{R + sL}{s^2 LC + sCR + 1} = \frac{2 + s \times 1}{s^2 1 \times 0.1 + s \times 0.1 \times 2 + 1} = \frac{s + 2}{0.1 s^2 + 0.2s + 1}$$

$$= \frac{10(s+2)}{s^2 + 2s + 10}$$

$$H(s) = Z = \frac{10(s+2)}{s^2 + 2s + 10}$$

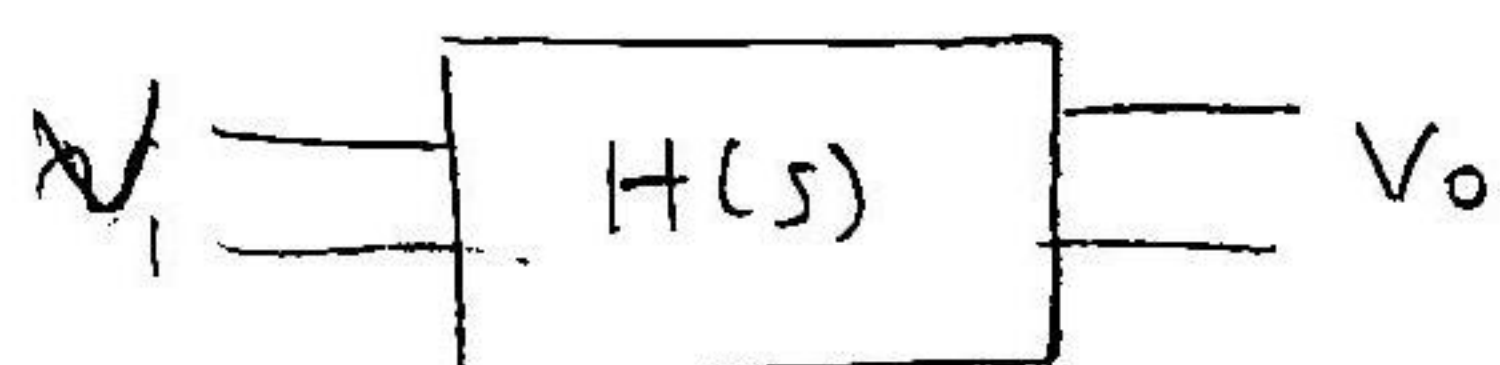
Poles $\rightarrow s^2 + 2s + 10 = 0$ $s_{1,2} = \frac{-2 \pm \sqrt{2^2 - 40}}{2} = -1 \pm 3j$

$$s_1 = -1 + 3j \quad s_2 = -1 - 3j$$

Zeros $\rightarrow 10(s+2) = 0 \Rightarrow s = -2$

Example problem: $H(s) = \frac{s+4}{s^2+3s+2}$

331



calculate $v_0(t)$ for

a) $v_1(t) = u(t)$ b) $v_1(t) = \cos 5t$

Solution

$$H(s) = \frac{V_0}{V_1} \Rightarrow V_0 = H(s) \cdot V_1$$

a) $v_1(t) = u(t) \rightarrow V_1(s) = \frac{1}{s}$ $V_0 = H(s) V_1(s) = \frac{1}{s} \frac{s+4}{s^2+3s+2}$

$$s^2+3s+2=0 \rightarrow s_1 = -2$$

$$\rightarrow s_2 = -1$$

$$V_0 = \frac{s+4}{s(s^2+3s+2)} = \frac{s+4}{s(s+2)(s+1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1}$$

$$A = 2 \quad B = 1 \quad C = -3$$

$$V_0(s) = \frac{2}{s} + \frac{1}{s+2} - \frac{3}{s+1} \rightarrow v_0(t) = 2u(t) + e^{-2t} - 3e^{-t}$$

b) $v_1(t) = \cos 5t \rightarrow V_1(s) = \frac{s}{s^2+5^2}$

$$V_0 = H(s) V_1(s) = \frac{s+4}{s^2+3s+2} \cdot \frac{s}{s^2+5^2} = \frac{s^2+4s}{(s+2)(s+1)(s+5j)(s-5j)}$$

$$= \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{s+5j} + \frac{D}{s-5j}$$

$$A = 0.1034$$

$$B = -0.077$$

$$C = -0.0133 + 0.113j$$

$$D = -0.0133 - 0.113j$$

$$V_0(s) = \frac{0.1034}{s+2} - \frac{0.077}{s+1} + \frac{-0.0133 + 0.113j}{s+5j} + \frac{-0.0133 - 0.113j}{s-5j}$$

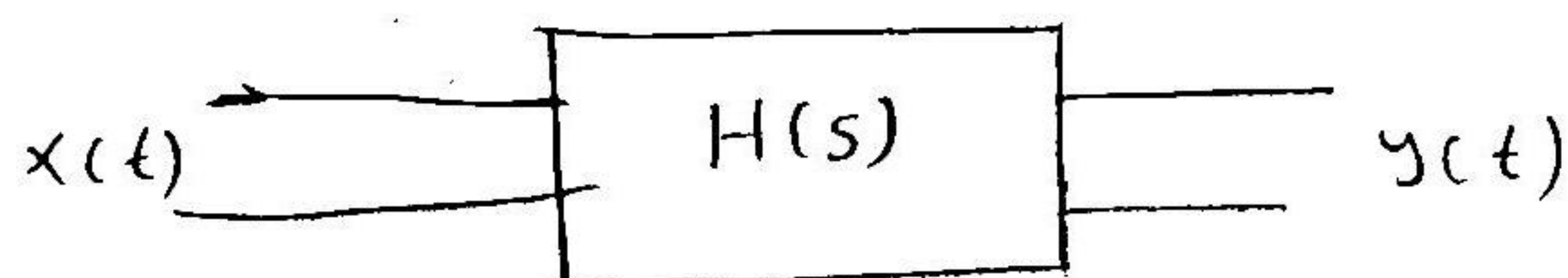
$$\mathcal{L} \left\{ \frac{a+bi}{s-(x+iy)} + \frac{a-bi}{s-(x-iy)} \right\} = e^{xt} \{ 2a \cos yt - 2b \sin yt \}$$

$$\mathcal{L} \left\{ \frac{-0.0133 + 0.113j}{s-(0-5j)} + \frac{-0.0133 - 0.113j}{s-(0+5j)} \right\} = e^0 (2(-0.013) \cos 5t - 2(0.11) \sin 5t)$$

$$v_0(t) = 0.1034 e^{-2t} - 0.077 e^{-t} - 2 \times 0.013 \cos 5t - 2 \times 0.11 \sin 5t$$

13.7 ♦ The Transfer Function and the Steady-State Sinusoidal Response

392



$$x(t) = A \cos(\omega t + \phi)$$

$$y(t) = A |H(j\omega)| \cos(\omega t + \phi + \theta(\omega))$$

$$\theta(\omega) = \angle H(j\omega)$$

Example problem: $H(s) = \frac{s+1}{s+2}$ $x(t) = 8 \cos 3t$ $y(t) = ?$

Solution

$$H(s) = \frac{s+1}{s+2} \rightarrow H(j\omega) = \frac{j\omega+1}{j\omega+2} \quad \omega=3 \rightarrow H(j\omega) = \frac{j3+1}{j3+2}$$

$$|H(j\omega)|_{\omega=3} = |H(j3)| = \frac{\sqrt{3^2+1^2}}{\sqrt{3^2+2^2}} = 0.95$$

$$\angle H(j\omega)_{\omega=3} = \angle H(j3) = \tan^{-1} \frac{3}{1} - \tan^{-1} \frac{3}{2} = 71.5 - 56.3 = 15.2$$

$$x(t) = 8 \cos 3t \rightarrow y(t) = 8 \times 0.95 \cos(3t + 15.2) = 7.6 \cos(3t + 15.2)$$

$$H(j\omega) = H(s) \Big|_{s=j\omega}$$

EXAMPLE 13.4

The circuit from Example 13.1 is shown in Fig. 13.46. The sinusoidal source voltage is $120 \cos(5000t + 30^\circ)$ V. Find the steady-state expression for v_o .

SOLUTION

From Example 13.1,

$$H(s) = \frac{1000(s+5000)}{s^2 + 6000s + 25 \times 10^6}$$

The frequency of the voltage source is 5000 rad/s; hence we evaluate $H(s)$ at $H(j5000)$:

$$\begin{aligned} H(j5000) &= \frac{1000(5000 + j5000)}{-25 \times 10^6 + j5000(6000) + 25 \times 10^6} \\ &= \frac{1+j1}{j6} = \frac{1-j1}{6} = \frac{\sqrt{2}}{6} \angle -45^\circ \end{aligned}$$

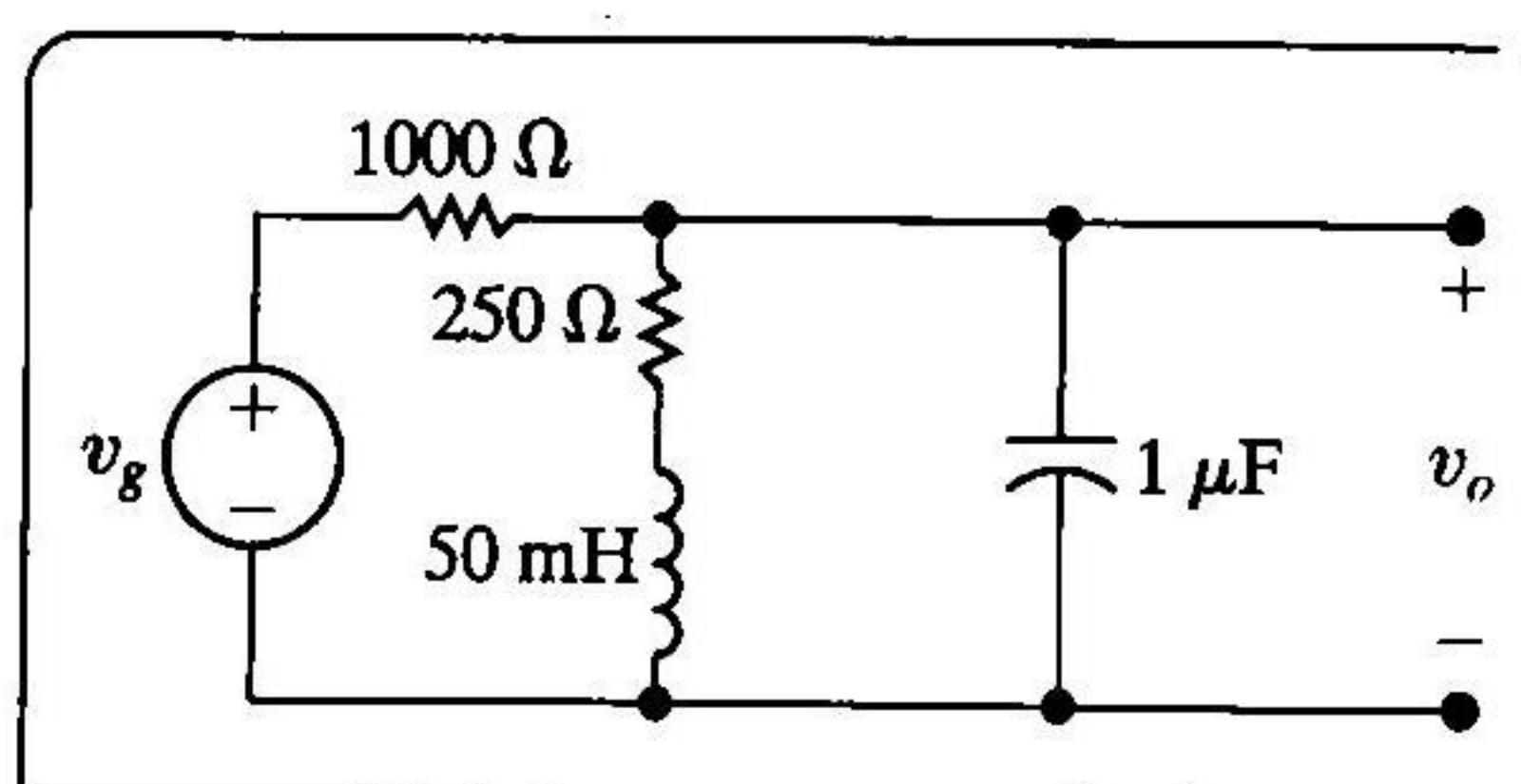


Figure 13.46 The circuit for Example 13.4.

Then, from Eq. 13.120,

$$\begin{aligned} v_{oss} &= \frac{(120)\sqrt{2}}{6} \cos(5000t + 30^\circ - 45^\circ) \\ &= 20\sqrt{2} \cos(5000t - 15^\circ) \text{ V} \end{aligned}$$