12.1 • Definition of the Laplace Transform

The Laplace transform of a function is given by the expression

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt, \qquad (12.1)$$

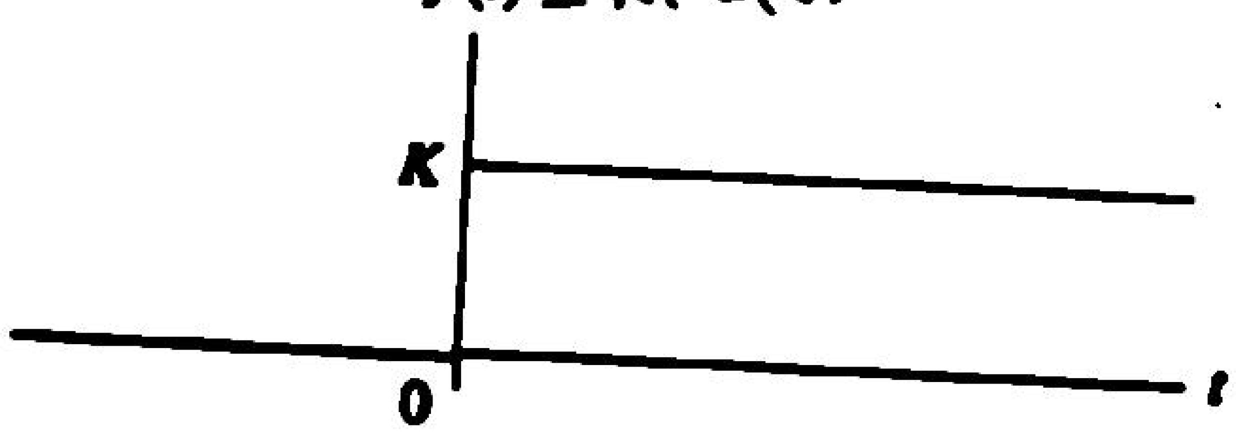
where the symbol $\mathcal{L}\{f(t)\}$ is read "the Laplace transform of $f(t)$."

The Laplace transform of $f(t)$ is also denoted $F(s)$; that is,

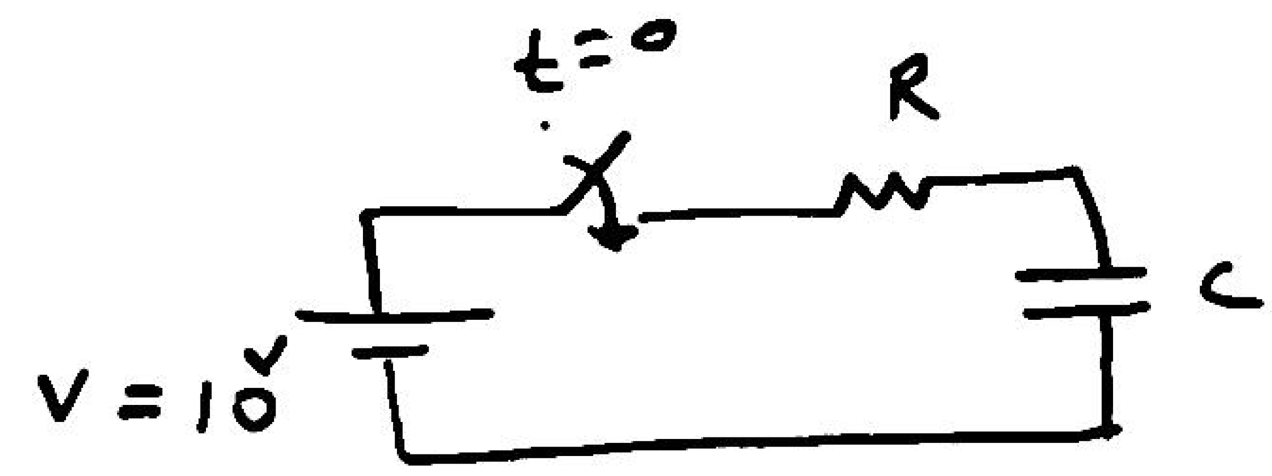
$$F(s) = \mathcal{L}\{f(t)\}. \qquad (12.2)$$

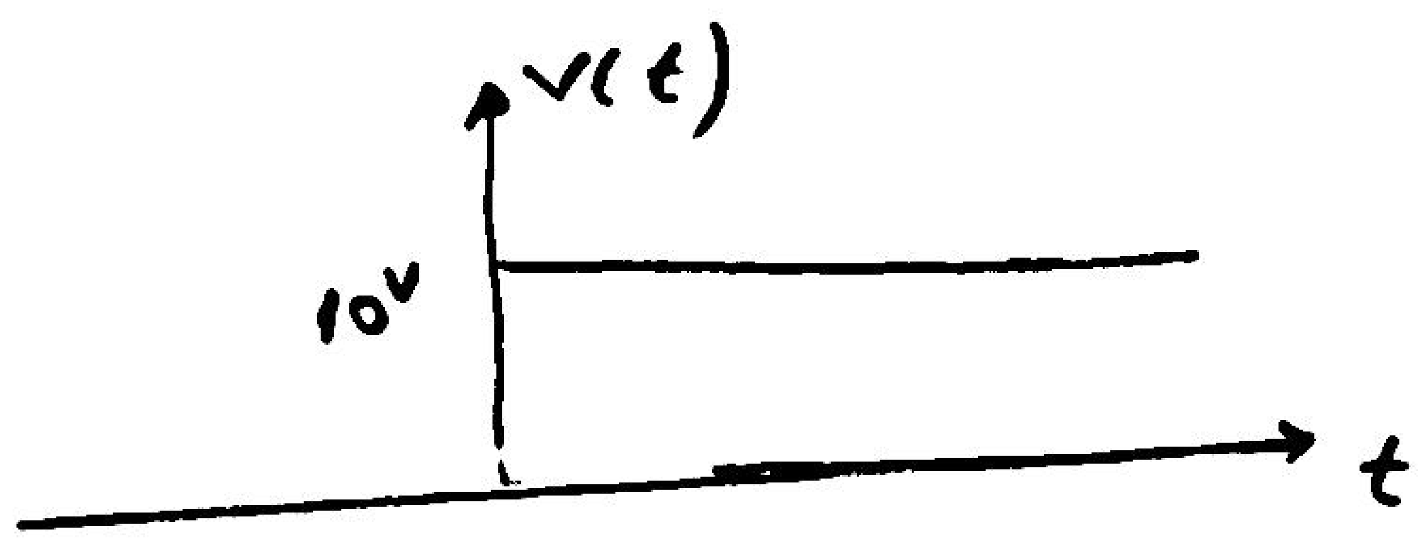
12.2 • The Step Function

$f(t) = K.\ u(t)$



[redacted] The step function.

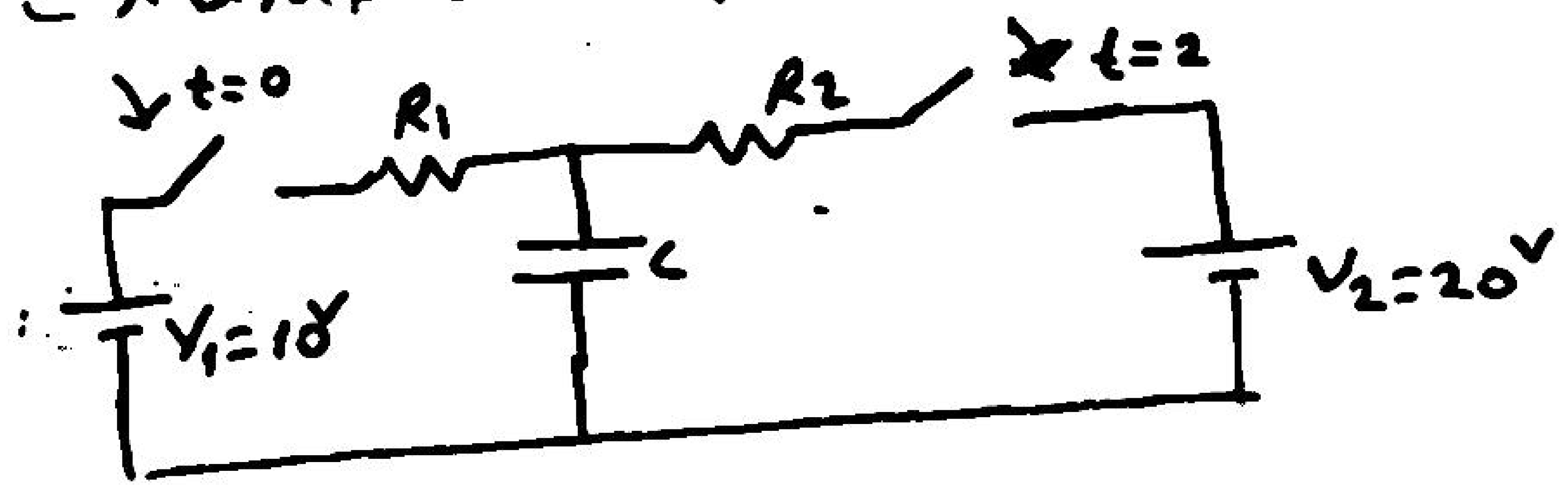


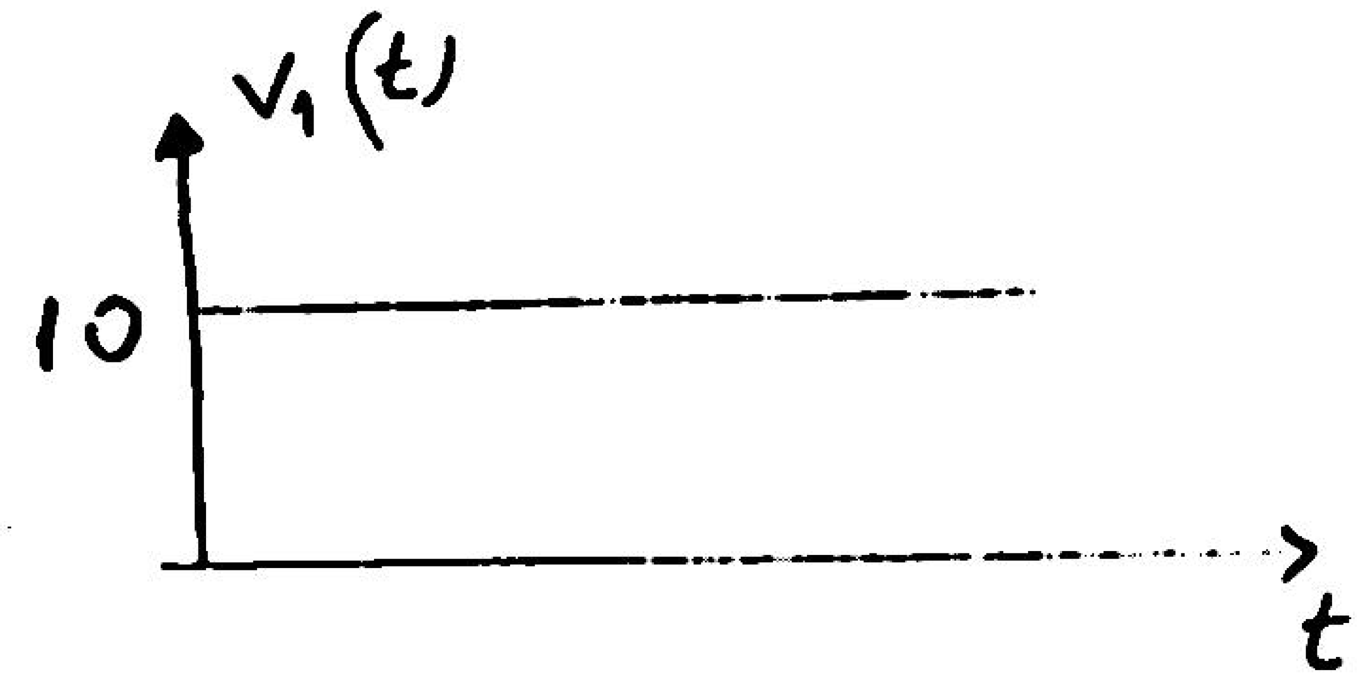


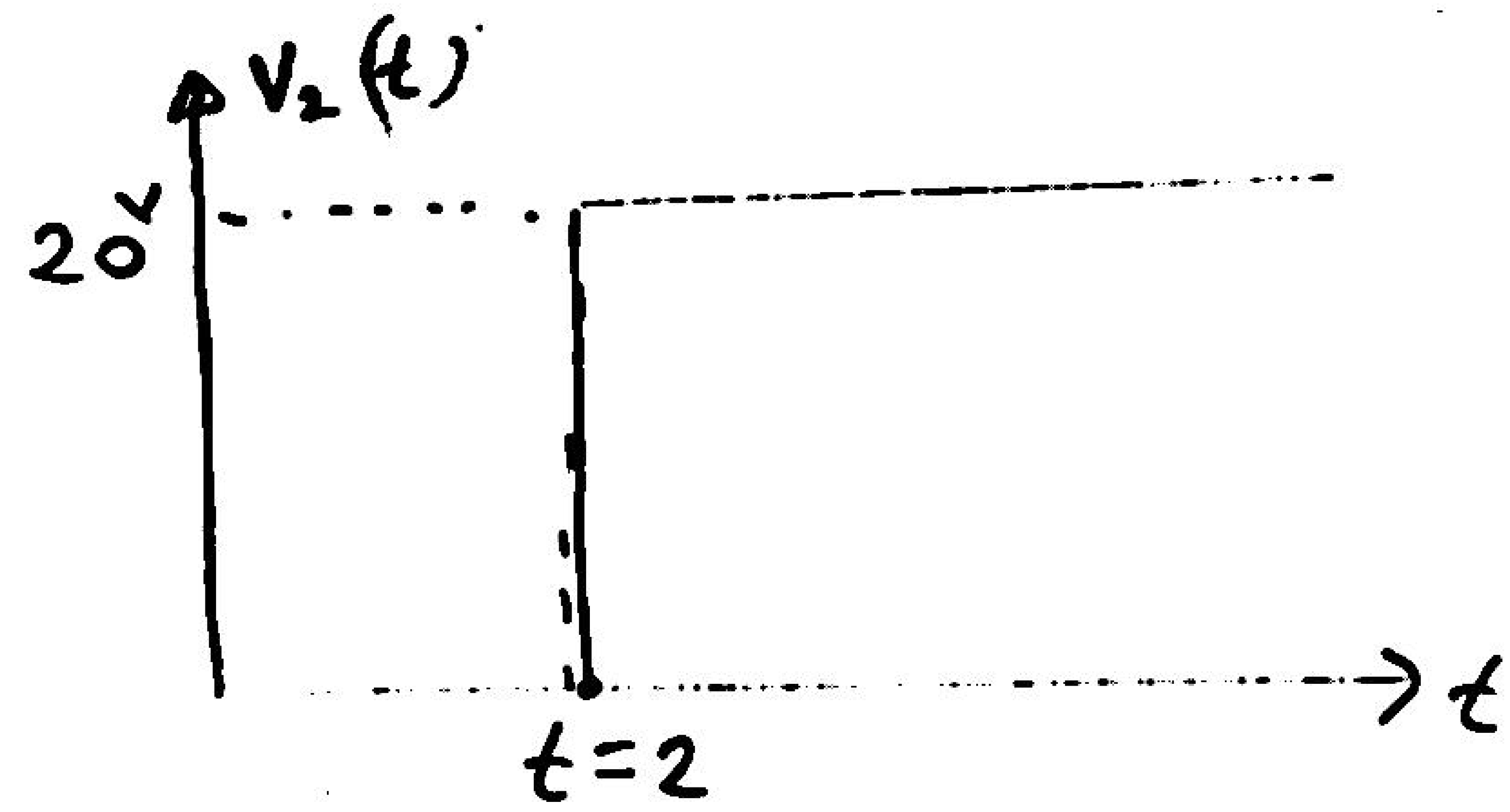
$v(t) = 0$ for $t < 0$

$v(t) = 10^V$ for $t \geq 0$

Example problem: at $t=0$ switch 1 is closed, at $t=2$ switch 2 is closed. Draw $V_1(t)$, $V_2(t)$



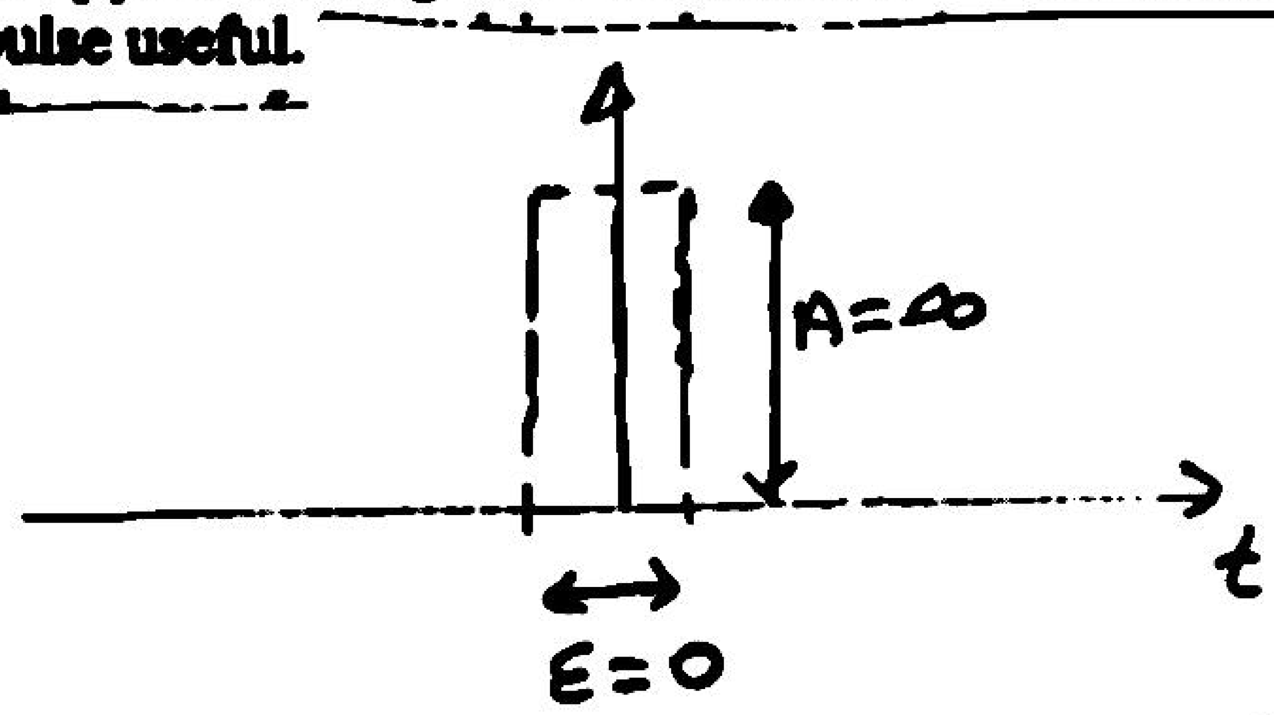




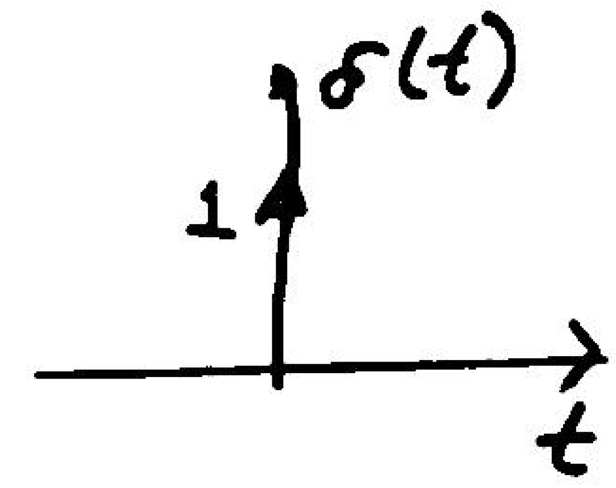
$V_2(t) = 0$ for $t < 2$

$V_2(t) = 20$ for $t \geq 2$

An **impulse** is a signal of infinite amplitude and zero duration. Such signals don't exist in nature, but some circuit signals come very close to approximating this definition, so we find a mathematical model of an impulse useful.



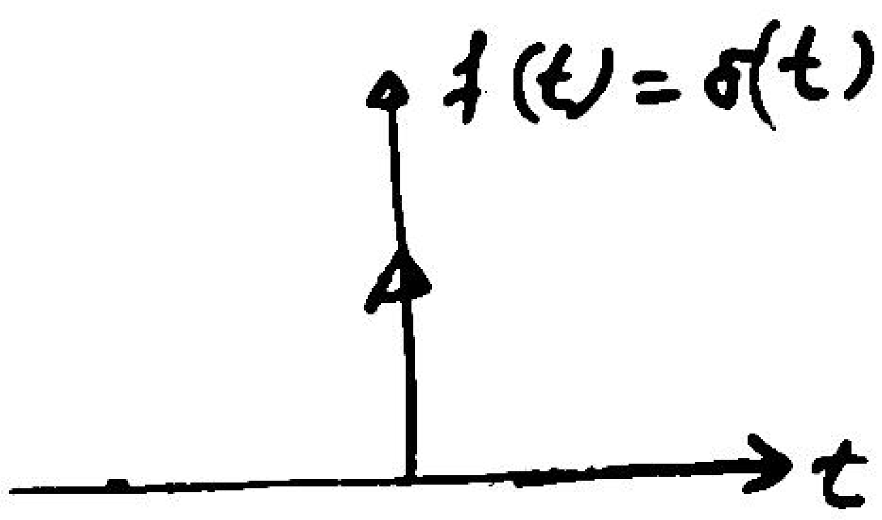
$A \times \varepsilon = \infty \times 0 = 1$



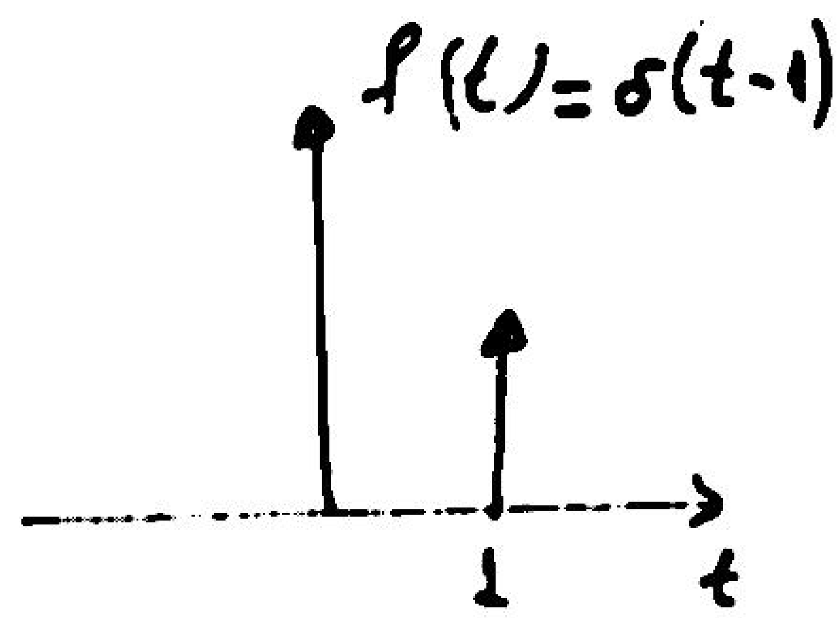
Example problem. show impulse function at

a) $t = 0$, b) $t = 1$ c) $t = 10$ d) $t = -2$

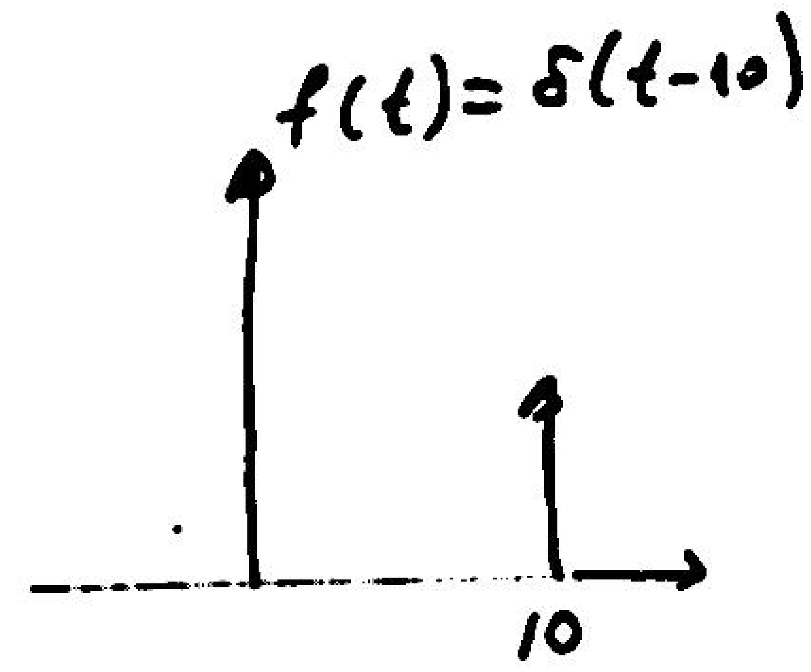
solution



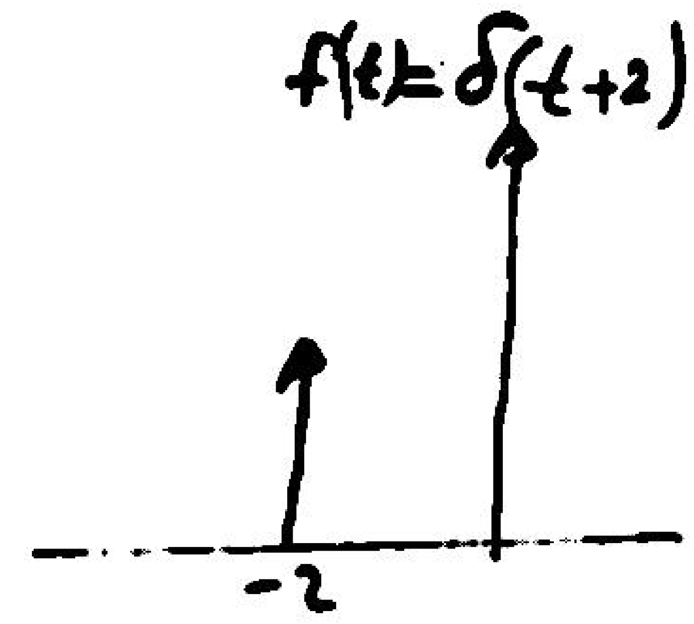
a)



b)

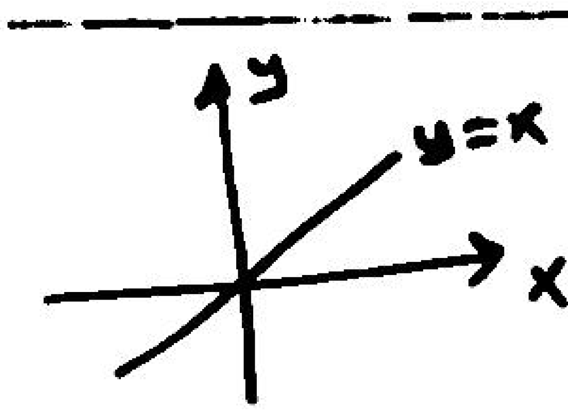


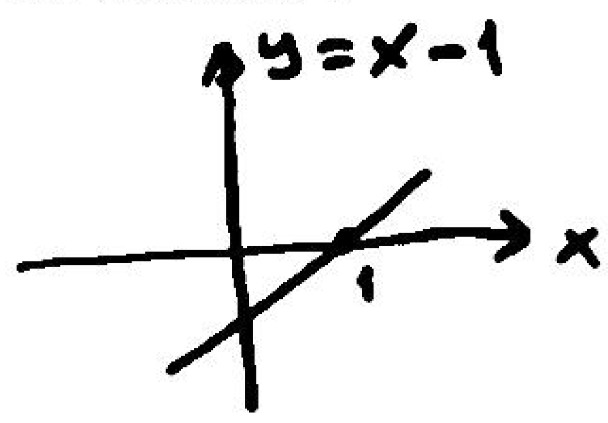
c)

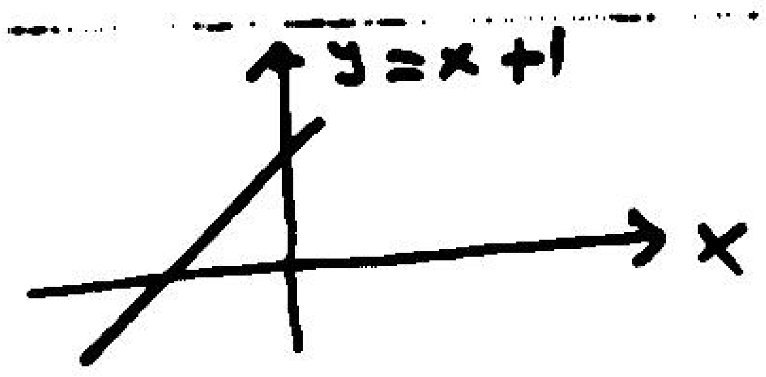


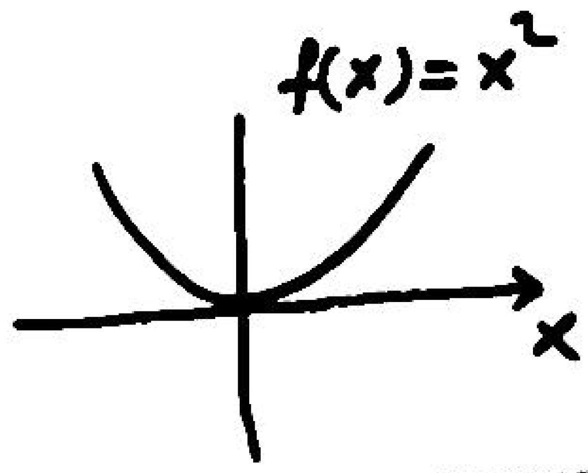
d)

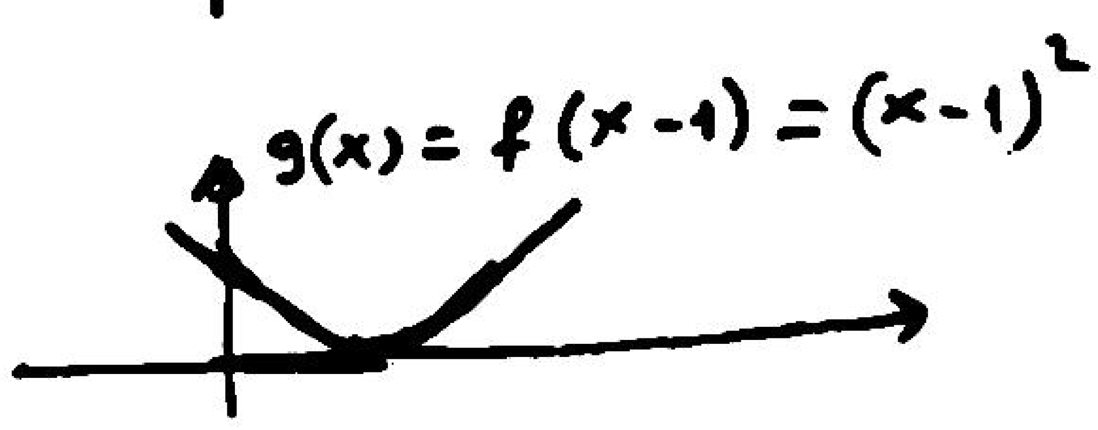
Note:

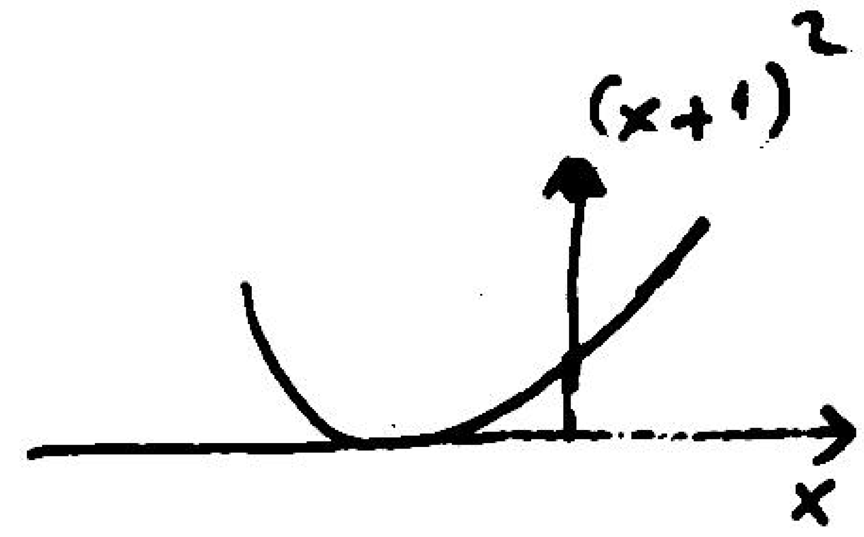










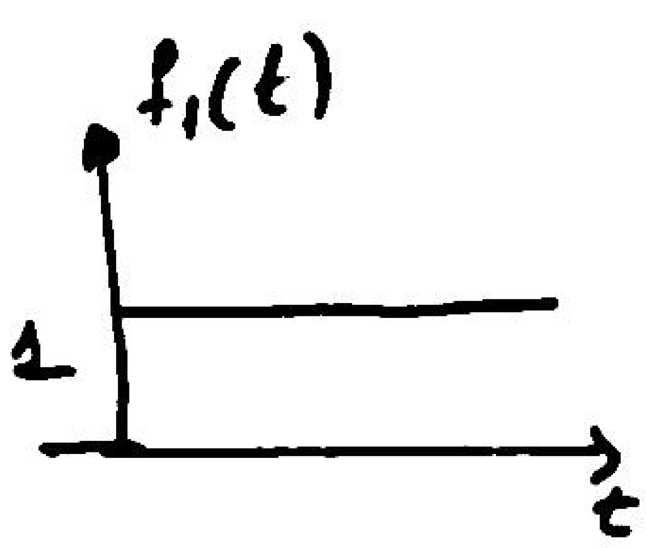


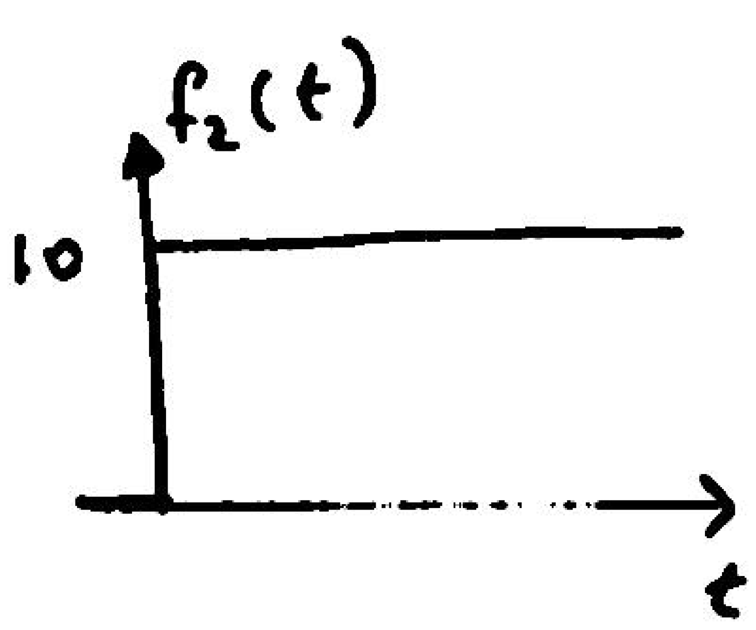
Example problem: Draw the following functions

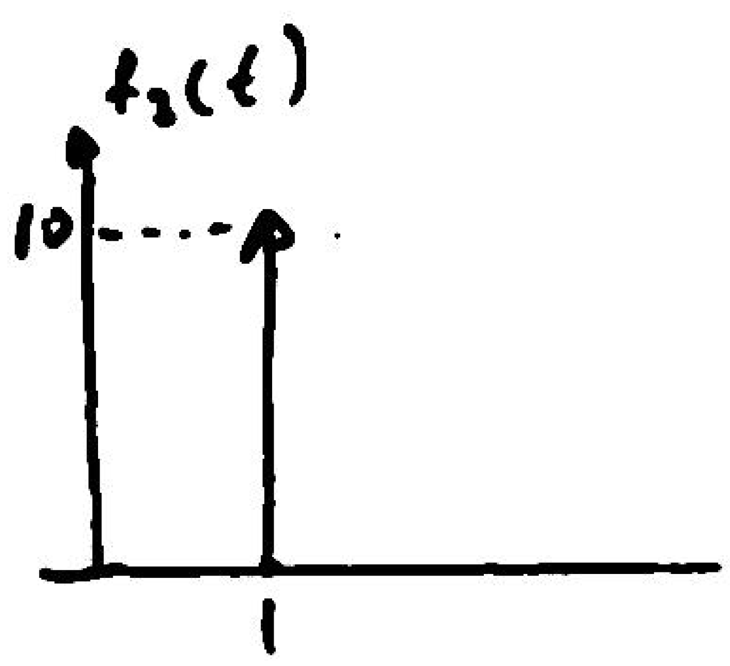
a) $f_1(t) = u(t)$ b) $f_2(t) = 10\ u(t)$ c) $f_3(t) = 10\,\delta(t-1)$

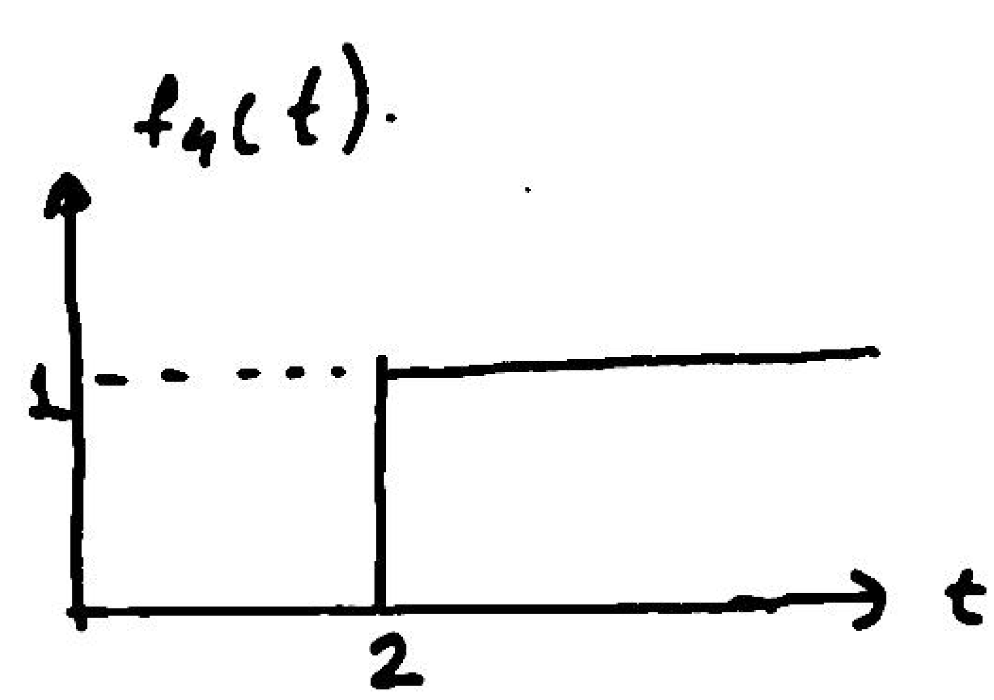
d) $f_4(t) = u(t-2)$

solution









Definition of impulse

$$\delta(t) = 0 \quad \text{for } t \neq 0$$

$$\delta(t) \neq 0 \quad \text{for } t = 0$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

$$\int_{-\infty}^{\infty} \delta(t-a)\, dt = 1$$

$$\int_{-\infty}^{\infty} \delta(t)\, f(t)\, dt = f(0)$$

$$\int_{-\infty}^{\infty} \delta(t-a)\, f(t)\, dt = f(a)$$

Example problem. Calculate the following integrals

$$S_1 = \int_{-\infty}^{\infty} \delta(t)\, t^2\, dt \qquad S_1 = t^2\Big|_{t=0} = 0^2 = 0$$

$$S_2 = \int_{-\infty}^{\infty} \delta(t-2)\, (t^2+5)\, dt \qquad S_2 = (t^2+5)\Big|_{t=2} = 2^2+5 = 9$$

$$S_3 = \int_{-\infty}^{\infty} \delta(t)\, e^{t^2+1}\, dt \qquad S_3 = e^{t^2+1}\Big|_{t=0} = e^1 = 2.71$$

$$S_4 = \int_{-\infty}^{\infty} \delta(t-3)\, \cos(2t+2)\, dt \qquad S_4 = \cos(2 \times 3 + 2) = -0.14$$

$\uparrow$ radian

Laplace Transform

Laplace Transform easies the solution of differential equations.

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

Laplace Transform exists if $f(t)$ is bounded.

$$\lim_{t\to\infty} f(t)\, e^{-st} < \infty$$

Example $f(t) = e^{at}$ $F(s) = ?$

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt = \int_0^\infty e^{at}\, e^{-st}\, dt = \int_0^\infty e^{(a-s)t}\, dt$$

$$= \frac{1}{a-s}\, e^{(a-s)t}\Big|_0^\infty = \frac{1}{a-s}\,(0-1) = \frac{-1}{a-s} = \frac{1}{s-a}$$

"$f(t)$ is bounded" means $e^{(a-s)\infty} \longrightarrow$ not ∞

<u>Result</u> $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

$\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$

Example calculate $\mathcal{L}\{\delta(t)\}$

$$F(s) = \int_0^\infty \delta(t)\, e^{-st}\, dt = e^0 = 1$$

Note: definition of dirac function is

$$\int_{-\infty}^{\infty} \delta(t)\, f(t) = f(0)$$

Example 3: $\mathcal{L}\{\cos wt\} = ?$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathcal{L}\{e^{jwt}\} = \frac{1}{s-jw}$$

$$\mathcal{L}\{e^{-jwt}\} = \frac{1}{s+jw}$$

$+$

$$\mathcal{L}\{e^{jwt} + e^{-jwt}\} = \frac{1}{s-jw} + \frac{1}{s+jw} = \frac{s+jw + s-jw}{(s-jw)(s+jw)} = \frac{2s}{s^2+w^2}$$

$$\cos w = \frac{1}{2}\{e^{jwt} + e^{-jwt}\}$$

$$\mathcal{L}\{\cos w\} = \mathcal{L}\{\tfrac{1}{2}(e^{jwt} + e^{-jwt})\} = \frac{1}{2}\,\frac{2s}{s^2+w^2} = \frac{s}{s^2+w^2}$$

$$\boxed{\mathcal{L}\{\cos wt\} = \frac{s}{s^2+w^2}}$$

Example $\mathcal{L}\{\sin wt\} = ?$

$$\mathcal{L}\{e^{jwt}\} = \frac{1}{s-jw}$$

$$\mathcal{L}\{e^{-jwt}\} = \frac{1}{s+jw}$$

$-$

$$\mathcal{L}\{e^{jwt} - e^{-jwt}\} = \frac{1}{s-jw} - \frac{1}{s+jw} = \frac{s+jw-(s-jw)}{(s-jw)(s+jw)} = \frac{2jw}{s^2+w^2}$$

$$\sin wt = \frac{1}{2j}\{e^{jwt} - e^{-jwt}\}$$

$$\mathcal{L}\{\sin wt\} = \frac{1}{2j}\mathcal{L}\{e^{jwt} - e^{-jwt}\} = \frac{1}{2j}\,\frac{2jw}{s^2+w^2} = \frac{w}{s^2+w^2}$$

$$\boxed{\mathcal{L}\{\sin wt\} = \frac{w}{s^2+w^2}}$$

Properties of Laplace Transform

OPERATION	$f(t)$	$F(s)$
Multiplication by a constant	$Kf(t)$	$KF(s)$
Addition/subtraction	$f_1(t) + f_2(t) - f_3(t) + \cdots$	$F_1(s) + F_2(s) - F_3(s) + \cdots$
First derivative (time)	$\frac{df(t)}{dt}$	$sF(s) - f(0^-)$
Second derivative (time)	$\frac{d^2 f(t)}{dt^2}$	$s^2F(s) - sf(0^-) - \frac{df(0^-)}{dt}$
nth derivative (time)	$\frac{d^n f(t)}{dt^n}$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}\frac{df(0^-)}{dt} - s^{n-3}\frac{df^2(0^-)}{dt^2} - \cdots - \frac{d^{n-1}f(0^-)}{dt^{n-1}}$
Time integral	$\int_0^t f(x)\,dx$	$\frac{F(s)}{s}$
Translation in time	$f(t-a)u(t-a), a > 0$	$e^{-as}F(s)$
Translation in frequency	$e^{-at}f(t)$	$F(s+a)$
Scale changing	$f(at), a > 0$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
First derivative (s)	$tf(t)$	$-\frac{dF(s)}{ds}$
nth derivative (s)	$t^n f(t)$	$(-1)^n\frac{d^nF(s)}{ds^n}$
s integral	$\frac{f(t)}{t}$	$\int_s^\infty F(u)\,du$

$$x(0^+) = \lim_{s\to\infty} s\mathcal{X}(s)$$

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} s\mathcal{X}(s)$$

$$\mathcal{L}\{x(t-t_0)\} = e^{-st_0}X(s)$$

Example problem 21: $f(t) = e^{at}\cos bt \quad F(s) = ?$

We know that $\mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}$

shiftin in s. domain $\mathcal{L}\{e^{\alpha t}x(t)\} = X(s-\alpha)$

$$\mathcal{L}\{e^{\alpha t}\cos bt\} = \left.\frac{s}{s^2+b^2}\right|_{s\to s-\alpha} = \frac{s-\alpha}{(s-\alpha)^2+b^2}$$

$$\mathcal{L}\{e^{-\alpha t}\cos bt\} = \frac{s+\alpha}{(s+\alpha)^2+b^2}$$

$$\mathcal{L}\{e^{-\alpha t}\sin bt\} = \left.\frac{b}{s^2+b^2}\right|_{s\to s+\alpha} = \frac{b}{(s+\alpha)^2+b^2}$$

Example problem: It is given that

$$\mathcal{L}\{\sin wt\} = \frac{w}{s^2+w^2}$$

Calculate $\mathcal{L}\{\cos wt\}$ by derivative theorem.

Solution

$$\mathcal{L}\{\frac{d}{dt}f(t)\} = s\,F(s) - f(0)$$

$$f(t) = \sin wt \Rightarrow \frac{df}{dt} = w\cos wt \Rightarrow \cos wt = \frac{1}{w}\frac{df}{dt}$$

$$\cos wt = \frac{1}{w}\frac{df}{dt}$$

$$\mathcal{L}\{\cos wt\} = \mathcal{L}\{\frac{1}{w}\frac{df}{dt}\} = \frac{1}{w}\mathcal{L}\{\frac{df}{dt}\}$$

$$= \frac{1}{w}\left[s\,F(s) - f(0)\right]$$

$$= \frac{1}{w}\left[s\left(\frac{w}{s^2+w^2}\right) - \sin w_x 0\right]$$

$$= \frac{s}{s^2+w^2}$$

Example problem: $\mathcal{L}\{\cos wt\} = \frac{s}{s^2+w^2}$ find $\mathcal{L}\{\sin wt\}$

Solution:

$$f(t) = \cos wt \qquad \frac{df}{dt} = -w\sin wt \qquad F(s) = \frac{s}{s^2+w^2}$$

$$f(0) = \cos 0 = 1$$

$$\sin wt = -\frac{1}{w}\frac{df}{dt} \Rightarrow \mathcal{L}\{\sin wt\} = -\frac{1}{w}\mathcal{L}\{\frac{df}{dt}\}$$

$$\mathcal{L}\{\sin wt\} = -\frac{1}{w}\{sF(s) - f(0)\} = -\frac{1}{w}\left[s\frac{s}{s^2+w^2} - 1\right] = -\frac{1}{w}\left[\frac{s^2}{s^2+w^2} - 1\right]$$

$$= -\frac{1}{w}\left[\frac{s^2}{s^2+w^2} - \frac{s^2+w^2}{s^2+w^2}\right] = -\frac{1}{w}\left[\frac{s^2-s^2-w^2}{s^2+w^2}\right] = \frac{w}{s^2+w^2}$$

Example problem: It is given that $\mathcal{L}\{e^{-2t}\} = \frac{1}{s+2}$

calculate $\lim_{t \to \infty} f(t)$ by <u>final value theorem</u>.

<u>Solution</u> $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$

$$f(t) = e^{-2t} \qquad F(s) = \frac{1}{s+2}$$

$$\lim_{s \to 0} sF(s) = \lim_{s \to 0} s\frac{1}{s+2} = 0 \times \frac{1}{0+2} = 0$$

Note: $\lim_{t \to \infty} f(t) = \lim_{t \to \infty} e^{-2t} = e^{-\infty} = 0$

Example problem: $\mathcal{L}\{1 - e^{-t}\} = \frac{1}{s} + \frac{1}{s+1}$ verify final value theorem.

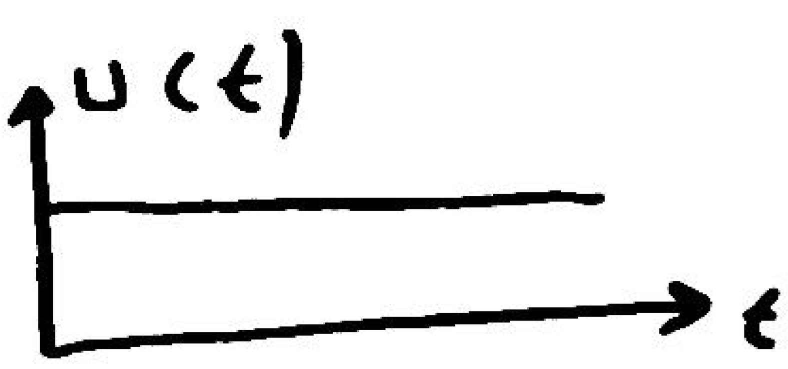
Solution

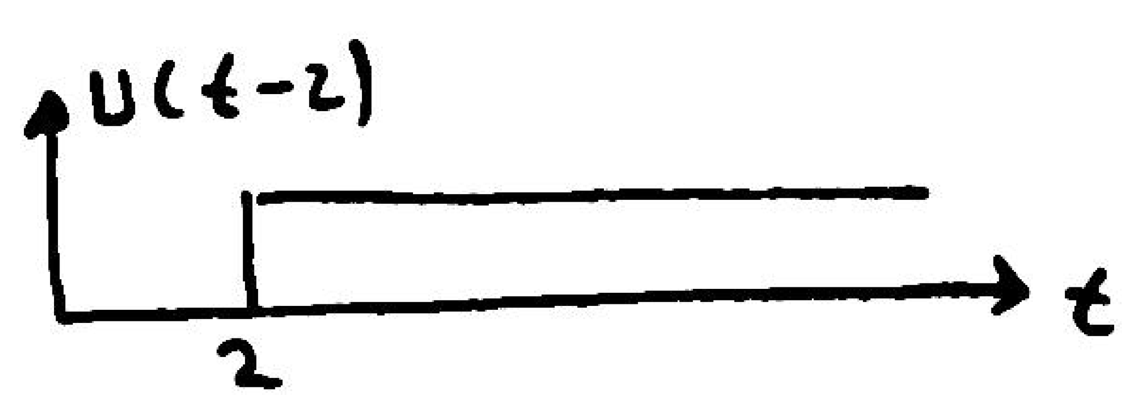
$$\lim_{t \to \infty} f(t) = \lim_{t \to \infty} (1 - e^{-t}) = 1 - e^{-\infty} = 1 - 0 = 1$$

$$\lim_{s \to 0} sF(s) = \lim_{s \to 0} s\left(\frac{1}{s} + \frac{1}{s+1}\right) = \lim_{s \to 0}\left(1 + \frac{s}{s+1}\right) = 1 + \frac{0}{0+1} = 1$$

Example problem: Draw $u(t-2)$ and calculate Laplace Tr.

<u>Solution.</u>





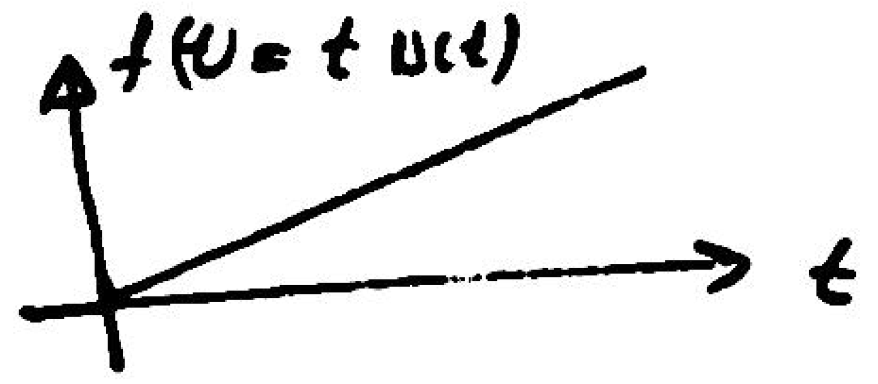
shifting theorem: $\mathcal{L}\{f(t-\alpha)\} = e^{-\alpha s} F(s)$

$$\mathcal{L}\{u(t)\} = \frac{1}{s}$$

$$\mathcal{L}\{u(t-2)\} = \frac{1}{s} \cdot e^{-2s}$$

Example Problem. Calculate $\mathcal{L}\{t\,u(t)\}$

Solution



Laplace transform formula

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$\mathcal{L}\{t\,u(t)\} = \int_0^\infty t \times 1\; e^{-st}\, dt =$$

Note: $\int x\, e^{ax}\, dx = e^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right)$

$$\int_0^\infty t\, e^{-st}\, dt = e^{-st}\left(\frac{t}{-s} - \frac{1}{(-s)^2}\right)\Bigg|_0^\infty = e^{-st}\left(-\frac{t}{s} - \frac{1}{s^2}\right)\Bigg|_0^\infty$$

$$= \left[e^{-\infty}\left(-\frac{\infty}{s} - \frac{1}{s^2}\right)\right] - \left[e^{-0}\left(-\frac{0}{s} - \frac{1}{s^2}\right)\right]_{t=0}$$

$$= 0 - \left(-\frac{1}{s^2}\right) = \frac{1}{s^2}$$

inverse Laplace Transform

1	$\delta(t)$
$\frac{1}{s}$	$u(t)$
$\frac{a}{s^2+a^2}$	$\sin at$
$\frac{s}{s^2+a^2}$	$\cos at$
$\frac{1}{s+a}$	e^{-at}

Example problem: Calculate inverse Laplace Trans

a) $f_1(s) = \frac{4}{s}$ b) $f_2(s) = 4 + \frac{5}{s}$ c) $f_3(s) = \frac{1}{s+6}$

d) $f_4(s) = \frac{8}{s+10}$ e) $f_5(s) = \frac{3}{s} + \frac{4}{s-2} - \frac{6}{s+3}$ f) $f_6(s) = \frac{1}{s^2+4}$

Solution

a) $f_1(s) = 4 \cdot \frac{1}{s} \Longrightarrow f_1(t) = 4\, u(t)$

b) $f_2(s) = 4 + \frac{5}{s} \Longrightarrow f_2(t) = 4\, \delta(t) + 5\, u(t)$

c) $f_3(s) = \frac{1}{s+6} \Longrightarrow f_3(t) = e^{-6t}$

d) $f_4(s) = \frac{8}{s+10} = 8\, \frac{1}{s+10} \Rightarrow f_4(t) = 8\, e^{-10t}$

e) $f_5(s) = \frac{3}{s} + 4\, \frac{1}{s-2} - 6\, \frac{1}{s+3} \Rightarrow f_5(t) = 3\, u(t) + 4\, e^{2t} - 6\, e^{-3t}$

f) $f_6(s) = \frac{1}{s^2+4} = \frac{1}{2}\, \frac{2}{s^2+2^2} \Rightarrow \frac{1}{2}\, \sin 2t$

Example problem: calculate $f(t)$

a) $F_1(s) = \frac{2}{s} + \frac{3}{s+1}$ b) $F_2(s) = \frac{3}{s+2} + \frac{4}{s^2+9}$

solution

a) $F_1(s) = 2 \cdot \frac{1}{s} + 3 \frac{1}{s+1} \Rightarrow f_1(t) = 2\,u(t) + 3\,e^{-t}$

b) $F_2(s) = 3 \frac{1}{s+2} + 4 \frac{1}{3} \frac{3}{s^2+3^2}$

$$\downarrow \qquad \downarrow$$

$$f_2(t) = 3\,e^{-2t} + \frac{4}{3} \sin 3t$$

Example problem calculate $f(t)$

$$F_1(s) = \frac{10\,s}{s^2+9} + \frac{8}{s^2+9} = 10 \frac{s}{s^2+3^2} + \frac{8}{3} \frac{3}{s^2+3^2}$$

$$f_1(t) = 10 \cos 3t + \frac{8}{3} \sin 3t$$

Example problem calculate $f(t)$

a) $F(s) = \frac{5s+4}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2} = \frac{A(s+2) + B\,s}{s(s+2)} = \frac{(A+B)\,s + 2A}{s(s+2)}$

$$\frac{5s+4}{s(s+2)} = \frac{(A+B)\,s + 2A}{s(s+2)}$$

$$\begin{pmatrix} 5 = A+B \\ 4 = 2A \end{pmatrix} \Rightarrow \begin{matrix} A = 2 \\ B = 3 \end{matrix}$$

$$F(s) = \frac{5s+4}{s(s+2)} = \frac{2}{s} + \frac{3}{s+2}$$

$$f(t) = 2\,u(t) + 3\,e^{-2t}$$

Example Problem calculate $f(t)$

$$f(s) = \frac{9s^2+52s+67}{s^3+9s^2+23s+15}$$

$$s^3+9s^2+23s+15=0 \Rightarrow \begin{array}{l} s_1=-1 \\ s_2=-3 \\ s_3=-5 \end{array}$$

$$\frac{9s^2+52s+67}{s^3+9s^2+23s+15} = \frac{9s^2+52s+67}{(s+1)(s+3)(s+5)} = \frac{A}{s+1}+\frac{B}{s+3}+\frac{C}{s+5}$$

calculate A, B, C

$A=2$

$B=3$

$C=4$

$$f(s) = \frac{9s^2+52s+67}{s^3+9s^2+23s+15} = \frac{2}{s+1}+\frac{3}{s+3}+\frac{4}{s+5}$$

$$f(t) = 2e^{-t}+3e^{-3t}+4e^{-5t}$$

Problem: How can we calculate A, B, C.

c~18

Residues: Partial Fraction Expansion.

$$f(z)=\frac{p(z)}{q(z)}=\frac{A}{z-a}+\frac{B}{z-b}+\frac{C}{z-c}.+\ldots\ldots$$

First residue Formula

$$A=\operatorname{Res}(z=a)=\lim_{z\to a}(z-a)\frac{p(z)}{q(z)}$$

$$B=\operatorname{Res}(z=b)=\lim_{z\to b}(z-b)\frac{p(z)}{q(z)}$$

$$C=\operatorname{Res}(z=c)=\lim_{z\to c}(z-c)\frac{p(z)}{q(z)}$$

Second residue Formula

$$A=\operatorname{Res}(z=a)=\lim_{z\to a}\frac{p(z)}{q'(z)}$$

$$B=\operatorname{Res}(z=b)=\lim_{z\to b}\frac{p(z)}{q'(z)}$$

$$C=\operatorname{Res}(z=c)=\lim_{z\to c}\frac{p(z)}{q'(z)}$$

Example AE-611

$$\frac{z+5}{z^3-5z^2-2z+24}=\frac{z+5}{(z-4)(z+2)(z-3)}$$

$$=\frac{A}{z-4}+\frac{B}{z+2}+\frac{C}{z-3}$$

$$A=\lim_{z\to4}(z-4)\frac{p(z)}{q(z)}=\lim_{z\to4}(z-4)\frac{z+5}{(z-4)(z-3)(z+2)}$$

$$=\lim_{z\to4}\frac{z+5}{(z-3)(z+2)}=\frac{4+5}{(4-3)(4+2)}=\frac{9}{6}=1.5$$

$$B=\lim_{z\to-2}(z-(-2))\frac{p(z)}{q(z)}=\lim_{z\to-2}(z+2)\frac{z+5}{(z-4)(z-3)(z+2)}$$

$$=\lim_{z\to-2}\frac{z+5}{(z-3)(z-4)}=\frac{-2+5}{(-2-3)(-2-4)}=\frac{3}{30}=0.1$$

$$C=\lim_{z\to3}(z-3)\frac{p(z)}{q(z)}=\lim_{z\to3}\frac{z+5}{(z-4)(z+2)}=\frac{3+5}{(3-4)(3+2)}=-1.6$$

Using the Second residue Formula

In our problem $p(z)=z+5$ $\quad q(z)=z^3-5z^2-2z+24$ and $q'(z)=3z^2-10z-2$

$$A=\lim_{z\to4}\frac{z+5}{3z^2-10z-2}=\frac{4+5}{3\ 4^2-10\ 4-2}=\frac{9}{6}=1.5$$

$$B=\lim_{z\to-2}\frac{z+5}{3z^2-10z-2}=\frac{-2+5}{3\ (-2)^2-10\ (-2)-2}=\frac{3}{30}=0.1$$

$$C=\lim_{z\to3}\frac{z+5}{3z^2-10z-2}=\frac{3+5}{3\ (3)^2-10\ (3)-2}=\frac{8}{-5}=$$

Note we get the same result by classical method

$$\frac{z+5}{z^3-5z^2-2z+24}=\frac{A}{z-4}+\frac{B}{z+2}+\frac{C}{z-3}$$

$$=\frac{A(z+2)(z-3)+B(z-4)(z-3)+C(z-4)(z+2)}{(z-4)(z+2)(z-3)}$$

$$=\frac{A(z^2-z-6)+B(z^2-7z-12)+C(z^2-2z-8)}{(z-4)(z+2)(z-3)}$$

$$=\frac{z^2(A+B+C)+z(-7A-B-2C)+12A-6B-8C}{(z-4)(z+2)(z-3)}$$

$$z^2(A+B+C)+z(-7A-B-2C)+12A-6B-8C=z+5$$

$A+B+C=0,\quad -7A-B-2C=1,\quad 12A-6B-8C=$

Solving for A,B,C we get A=1.5 B=0.1 C=-1.6

$$\frac{z+5}{z^3-5z^2-2z+24}=\frac{1.5}{z-4}+\frac{0.1}{z+2}+\frac{-1.6}{z-3}$$

Example AE-612

$$\frac{2z+12}{z^2+2z+2}=\frac{2z+12}{[z-(-1+i)][z-(-1-i)]}=\frac{2z+12}{(z+1-i)(z+1+}$$

$$=\frac{A}{(z+1-i)}+\frac{B}{(z+1+i)}$$

Using the first residue formula

$$A=\lim_{z\to-1+i}(z+1-i)\frac{2z+12}{(z+1-i)(z+1+i)}=\lim_{z\to-1+i}\frac{2z+12}{(z+1+i)}$$

$$=\frac{2(-1+i)+12}{((-1+i)+1+i)}=\frac{2i+10}{2i}=\frac{2i}{2i}+\frac{10}{2i}=1-5i$$

$$B=\lim_{z\to-1-i}(z+1+i)\frac{2z+12}{(z+1-i)(z+1+i)}=\lim_{z\to-1-i}\frac{2z+12}{(z+1-i)}=1+5i$$

Using the second residue formula

$p(z)=2z+12,\quad q(z)=z^2+2z+2,\quad q'(z)=2z+2$

$$A=\lim_{z\to-1+i}\frac{p(z)}{q'(z)}=\frac{2z+12}{2z+2}=\frac{2(-1+i)+12}{2(-1+i)+2}=1-5i$$

$$B=\lim_{z\to-1-i}\frac{p(z)}{q'(z)}=\frac{2z+12}{2z+2}=\frac{2(-1-i)+12}{2(-1-i)+2}=1+5i$$

Using classical Method

$$\frac{2z+12}{z^2+2z+2}=\frac{A}{(z+1-i)}+\frac{B}{(z+1+i)}=\frac{A(z+1+i)+B(z+1-i)}{(z+1-i)(z+1+i)}$$

(A+B)=2 $\quad$ A(1+i)+B(1-i)=12 $\quad$ solution is A=1-5i , B=1+5i

Result $$\frac{2z+12}{z^2+2z+2}=\frac{1+5i}{(z+1-i)}+\frac{1-5i}{(z+1+i)}$$

EX.- 28 $\quad f(s) = \frac{s+3}{(s+1)(s+2)} = \frac{s+3}{s^2+3s+2} \quad f(t) = ?$ 335

<u>solution</u>

$$\frac{s+3}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$A = (s+1)\, f(s)\Big|_{s=-1} = (s+1)\,\frac{s+3}{(s+1)(s+2)}\Big|_{s=-1} = \frac{s+3}{s+2}\Big|_{s=-1} = \frac{-1+3}{-1+2} = 2$$

$$\boxed{A = 2}$$

$$B = (s+2)\, f(s)\Big|_{s=-2} = (s+2)\,\frac{s+3}{(s+1)(s+2)}\Big|_{s=-2} = \frac{s+3}{s+1}\Big|_{s=-2} = \frac{-2+3}{-2+1} = -1$$

$$B = -1$$

$$f(s) = \frac{s+3}{(s+1)(s+2)} = \frac{2}{s+1} + \frac{-1}{s+2}$$

$$f(t) = 2e^{-t} - e^{-2t}$$

complex roots:

$$\frac{As+B}{s^2+ms+n} = \frac{c_1}{s-s_1} + \frac{c_2}{s-s_2}$$

c_1, c_2, s_1, s_2 are complex and $c_2 = c_1^*$ $\quad S_2 = s_1^*$

Examples

$$\frac{10s+14}{s^2+6s+14} = \frac{5+2i}{s-(-3+4i)} + \frac{5-2i}{s-(-3-4i)}$$

$$\frac{6s+2}{s^2+6s+14} = \frac{3+2i}{s-(-3+4i)} + \frac{3-2i}{s-(-3-4i)}$$

$$\frac{4s-22}{s^2+10s+34} = \frac{2+7i}{s-(-5+3i)} + \frac{2-7i}{s-(-5-3i)}$$

$$\frac{4s+62}{s^2+10s+34} = \frac{2-7i}{s-(-5+3i)} + \frac{2+7i}{s-(-5-3i)}$$

$$\frac{As+B}{s^2+ms+n} = \frac{a+bi}{s-(x+iy)} + \frac{a-bi}{s-(x-iy)}$$

$$\mathcal{L}^{-1}\left\{\frac{a+bi}{s-(x+iy)} + \frac{a-bi}{s-(x-iy)}\right\} = (a+bi)\,e^{(x+iy)t} + (a-bi)\,e^{(x-iy)t}$$

$$= (a+bi)\,e^{xt}e^{iyt} + (a-bi)\,e^{xt}\,e^{-iyt} = e^{xt}\left((a+bi)e^{iyt} + (a-bi)\,e^{-iyt}\right)$$

$$= e^{xt}\left[(a+bi)(\cos yt + i\sin yt) + (a-bi)(\cos yt - i\sin yt)\right]$$

$$= e^{xt}\left[a\cos yt + ai\sin yt + bi\cos yt - b\sin yt + a\cos yt - ai\sin yt - bi\cos yt - b\sin yt\right]$$

$$\mathcal{L}^{-1}\left\{ \frac{a+bi}{s-(x+iy)} + \frac{a-bi}{s-(x-iy)} \right\} = e^{xt}\left[2a\cos yt - 2b\sin yt \right]$$

$$\mathcal{L}^{-1}\left\{ \frac{10s+14}{s^2+6s+25} \right\} = \mathcal{L}^{-1}\left\{ \frac{5+2i}{s-(-3+4i)} + \frac{5-2i}{s-(-3-4i)} \right\} = e^{-3t}\left(2\times 5\cos 4t - 2\times 2\sin 4t\right)$$

$$= e^{-3t}\left(10\cos 4t - 4\sin 4t\right)$$

$$\mathcal{L}^{-1}\left\{ \frac{6s+2}{s^2+6s+25} \right\} = \mathcal{L}^{-1}\left\{ \frac{3+2i}{s-(-3+4i)} + \frac{3-2i}{s-(-3-4i)} \right\} = e^{-3t}\left(2\times 3\cos 4t - 2\times 2\sin 4t\right)$$

$$= e^{-3t}\left(6\cos 4t - 4\sin 4t\right)$$

$$\mathcal{L}^{-1}\left\{ \frac{4s-22}{s^2+10s+34} \right\} = \mathcal{L}^{-1}\left\{ \frac{2+7i}{s-(-5+3i)} + \frac{2-7i}{s-(-5-3i)} \right\} = e^{-5t}\left(2\times 2\cos 3t - 2\times 7\sin 3t\right)$$

$+3i$ $+7i$

$$= e^{-5t}\left(4\cos 3t - 14\sin 3t\right)$$

$$\mathcal{L}^{-1}\left\{ \frac{4s+62}{s^2+10s+34} \right\} = \mathcal{L}^{-1}\left\{ \frac{2-7i}{s-(-5+3i)} + \frac{2+7i}{s-(-5-3i)} \right\} = e^{-5t}\left(2\times 2\cos 3t - 2\times(-7)\sin 3t\right)$$

$+3i$ $-7i$

$$= e^{-5t}\left(4\cos 3t + 14\sin 3t\right)$$

$$\mathcal{L}^{-1}\left\{ \frac{4s+22}{s^2-10s+34} \right\} = \mathcal{L}^{-1}\left\{ \frac{2-7i}{s-(5+3i)} + \frac{2+7i}{s-(5-3i)} \right\} = e^{5t}\left(2\times 2\cos 4t - 2\times(-7)\sin 4t\right)$$

$$= e^{5t}\left(4\cos 4t + 14\sin 4t\right)$$

$$\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathcal{L}\{e^{-at}\cos bt\} = \frac{s+a}{(s+a)^2+b^2}$$

$$\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2} \qquad \mathcal{L}\{e^{-at}\sin bt\} = \frac{b}{(s+a)^2+b^2}$$

EX-34

$$\frac{10s+14}{s^2+6s+25} \qquad s^2+6s+25=0 \to \begin{cases} s_1=-3+4i \\ s_2=-3-4i \end{cases}$$

$$s^2+6s+25=(s+3)^2+4^2$$

$$14 = 10 \times 3 + x$$
$$x = 14-30 = -16$$

$$\frac{10s+14}{s^2+6s+25} = \frac{10s+14}{(s+3)^2+4^2} = 10\frac{(s+3)}{(s+3)^2+4^2} + \frac{-16}{(s+3)^2+4^2}$$

$$= 10\,\frac{s+3}{(s+3)^2+4^2} + (-4)\,\frac{4}{(s+3)^2+4^2}$$

$$\downarrow$$

$$10\,e^{-3t}\cos 4t + (-4)\,e^{-3t}\sin 4t$$

$$= e^{-3t}(10\cos 4t - 4\sin 4t)$$

EX-35

$$2 = 6 \times 3 + x$$
$$2-18 = x$$

$$\frac{6s+2}{s^2+6s+25} = \frac{6s+2}{(s+3)^2+4^2} = 6\,\frac{s+3}{(s+3)^2+4^2} + \frac{-16}{(s+3)^2+4^2} = 6\,\frac{s+3}{(s+3)^2+4^2} - 4\,\frac{4}{(s+3)^2+4^2}$$

$$\downarrow$$

$$6\,e^{-3t}\cos 4t - 4e^{-3t}\sin 4t$$

$$= e^{-3t}(6\cos 4t - 4\sin 4t)$$

EX-36

$$\mathcal{L}^{-1}\left\{\frac{4s-22}{s^2+10s+34}\right\}=?$$

$$s^2+10s+34=0 \quad \begin{matrix} \nearrow s_1=-5+3i \\ \searrow s_2=-5-3i \end{matrix}$$

$-22=4\times 5+x$

$-22-20=x$

$$\frac{4s-22}{s^2+10s+34}=\frac{4s-22}{(s+5)^2+3^2}=4\,\frac{s+5}{(s+5)^2+3^2}+\frac{-42}{(s+5)^2+3^2}$$

$$=4\,\frac{s+5}{(s+5)^2+3^2}-14\,\frac{3}{(s+5)^2+3^2}$$

$$\swarrow$$

$$4e^{-5t}\cos 3t-14e^{-5t}\sin 3t$$

$$=e^{-5t}(4\cos 3t-14\sin 3t)$$

EX-37

$$\mathcal{L}^{-1}\left\{\frac{4s+22}{s^2-10s+34}\right\}$$

$$s^2-10s+34=0 \quad \begin{matrix} \nearrow s_1=5+3i \\ \searrow s_2=5-3i \end{matrix}$$

$22=4\times(-5)+x$

$22+20=x$

$$\frac{4s+22}{s^2-10s+34}=\frac{4s+22}{(s-5)^2+3^2}=4\,\frac{s-5}{(s-5)^2+3^2}+\frac{42}{(s-5)^2+3^2}$$

$$=4\,\frac{s-5}{(s-5)^2+3^2}+14\,\frac{3}{(s-5)^2+3^2}$$

$$\swarrow$$

$$4e^{5t}\cos 3t+14e^{5t}\sin 3t$$

$$=e^{5t}(4\cos 3t+14\sin 3t)$$

Ex-41

s-345

$$F(s)=\frac{13s^2+82s+122}{s^3+9s^2+26s+24}=\frac{1}{s+4}+\frac{7}{s+3}+\frac{5}{s+2}$$

$$\downarrow$$

$$f(t)=e^{-4t}+7e^{-3t}+5e^{-2t}$$

Ex-42

$$F(s)=\frac{17s^4-16s^3+497s^2-1200s+10978}{s^5-4s^4+23s^3-416s^2+2966s-4420}$$

$$=\underbrace{\frac{3}{s-(-4+7i)}+\frac{3}{s-(-4-7i)}}+\underbrace{\frac{2-7i}{s-(5+3i)}+\frac{2+7i}{s-(5-3i)}}+\frac{7}{s-2}$$

$$f(t)=e^{-4t}(2\times3\cos 7t-2\times0\sin 3t)+e^{5t}(2\times2\cos 3t-2\times(-7)\sin 3t)+7e^{2t}$$

$$=6e^{-4t}\cos 7t+e^{5t}(4\cos 3t+14\sin 3t)+7e^{2t}$$

Repeated real roots

$$F(s) = \frac{10}{(s+2)} \longrightarrow f(t) = 10\, e^{-2t}$$

$$f(s) = \frac{10}{(s+2)^2} \longrightarrow f(t) = 10\, t\, e^{-2t}$$

$$f(s) = \frac{10}{(s+2)^3} \longrightarrow f(t) = 10\, t^2\, e^{-2t}$$

$$F(s) = \frac{2}{s+4} + \frac{3}{(s+4)^2} \longrightarrow f(t) = 2\, e^{-4t} + 3\, t\, e^{-4t}$$

$$f(s) = \frac{2}{s+2} + \frac{3}{(s+4)^2} + \frac{5}{(s+1)^3} \rightarrow f(t) = 2\, e^{-2t} + 3\, t\, e^{-4t} + 5\, t^2 e^{-t}$$

$$f(s) = \frac{1}{s} + \frac{2}{s^2} + \frac{3}{s+1} \longrightarrow f(t) = u(t) + t\, u(t) + 3\, e^{-t}$$

$$f(s) = \frac{100(s+25)}{s(s+5)^3} = \frac{A}{s} + \frac{B}{(s+5)} + \frac{C}{(s+5)^2} + \frac{D}{(s+5)^2}$$

$A = 20 \qquad B = -400 \qquad C = -100 \qquad D = -20$

$$F(s) = \frac{20}{s} - \frac{400}{s+5} - \frac{100}{(s+5)^2} - \frac{20}{(s+5)^3}$$

$$f(t) = 20\, u(t) - 400\, e^{-5t} - 100\, t\, e^{-5t} - 20\, t^2 e^{-5t}$$

How can we calculate A, B, C, D

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$$F(s) = \frac{ms+n}{(s+a)^2(s+b)} = \frac{A}{s+a} + \frac{B}{(s+a)^2} + \frac{C}{s+b}$$

$$C = (s+b)\,F(s)\Big|_{s=-b} = (s+b)\frac{ms+n}{(s+a)^2(s+b)}\Bigg|_{s=-b} = \frac{ms+n}{(s+a)^2}\Bigg|_{s=-b}$$

$$= \frac{m(-b)+n}{(-b+a)^2}$$

$$B = (s+a)^2\,F(s)\Big|_{s=-a} = (s+a)^2\frac{ms+n}{(s+a)^2(s+b)}\Bigg|_{s=-a} = \frac{ms+n}{s+b}\Bigg|_{s=-a} = \frac{-ma+n}{-a+b}$$

$$A = \frac{d}{ds}\left[(s+a)^2 F(s)\right]\Bigg|_{s=-a} = \frac{d}{ds}\left[\frac{ms+n}{s+b}\right] = \frac{(ms+n)'(s+b)-(ms+n)(s+b)'}{(s+b)^2}$$

$$= \frac{m(s+b)-(ms+n)}{(s+b)^2} =$$

$$F(s) = \frac{6s^2+17s+14}{(s+1)^2(s+2)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+2}$$

$$C = \frac{6s^2+17s+14}{(s+1)^2}\Bigg|_{s=-2} = \frac{6\times(-2)^2+17(-2)+14}{(-2+1)^2} = 4$$

$$B = \frac{6s^2+17s+14}{s+2}\Bigg|_{s=-1} = \frac{6(-1)^2+17(-1)+14}{-1+2} = 3$$

$$A = \frac{d}{ds}\left[\frac{6s^2+17s+14}{s+2}\right]\Bigg|_{s=-1} = \frac{(12s+17)(s+2)-1(6s^2+17s+14)}{(s+2)^2}\Bigg|_{s=-1} = 2$$

$$\frac{15z^3+1}{(z+2)^3(z-1)^2(z+10)} = \frac{A_1}{(z+2)} + \frac{A_2}{(z+2)^2} + \frac{A_3}{(z+2)^3} + \frac{B_1}{(z-1)} + \frac{B_2}{(z-1)^2} + \frac{C}{(z+10)}$$

$$C = (z+10)F(z)\Big|_{z=-10} = \frac{15z^3+1}{(z+2)^3(z-1)^2}\Bigg|_{z=-10}$$

$$= \frac{15(-10)^3+1}{(-10+2)^3(-10-1)^2} = 0.2421$$

$$B_2 = (z-1)^2F(z)\Big|_{z=1} = \frac{15z^3+1}{(z+2)^3(z+10)}\Bigg|_{z=1}$$

$$= \frac{15(1)^3+1}{(1+2)^3(1+10)} = 0.0539$$

$$B_1 = \frac{d}{dz}\left[(z-1)^2F(z)\right]\Big|_{z=1} = \frac{d}{dz}\left[\frac{15z^3+1}{(z+2)^3(z+10)}\right]_{z=1}$$

$p=15z^3+1$ $\quad p'=45z^2$

$q=(z+2)^3(z+10)$ $\quad q'=3(z+2)^2(z+10)+(z+2)^3\,1$

$q'=4z^3+48z^2+144z+128$

$$\left(\frac{p}{q}\right)' = \left(\frac{p'q-pq'}{q^2}\right)$$

$$= \frac{-15z^6+1080z^4+3836z^3+3552z^2-144z-128}{(z+2)^6(z+10)^2}$$

$$\frac{-15z^6+1080z^4+3836z^3+3552z^2-144z-128}{(z+2)^6(z+10)^2}\Bigg|_{z=1}$$

$$\frac{-15\,1^6+1080\,1^4+3836\,1^3+3552\,1^2-144\,1-128}{(1+2)^6(1+10)^2} = 0.092$$

$B_1 = 0.092$

$$A_3 = (z+2)^3F(z)\Big|_{z=-2} = \frac{15z^3+1}{(z-1)^2(z+10)}\Bigg|_{z=-2}$$

$$= \frac{15(-2)^3+1}{(-2-1)^2(-2+10)} = -1.6528$$

$$A_2 = \frac{d}{dz}\left[(z+2)^3F(z)\right]\Big|_{z=-2} = \frac{d}{dz}\left[\frac{15z^3+1}{(z-1)^2(z+10)}\right]_{z=-2}$$

$p=15z^3+1$ $\quad p'=45z^2$

$q=(z-1)^2(z+10)$ $\quad q'=2(z-1)(z+10)+(z-1)^2\,1$

$q'=3z^2+16z-19$

$A_2 = 1.604$

$$A_1 = \frac{d^2}{dz^2}\left[\frac{15z^3+1}{(z-1)^2(z+10)}\right]\Bigg|_{z=-2} = -0.33$$

$$\frac{15z^3+1}{(z+2)^3(z-1)^2(z+10)} =$$

$$\frac{-0.33}{z+2} + \frac{1.604}{(z+2)^2} + \frac{-1.652}{(z+2)^3} + \frac{0.092}{z-1}$$

$$+ \frac{0.0539}{(z-1)^2} + \frac{0.2421}{z+10}$$

Improper Rational functions

$$F(s) = \frac{s^2+7s+5}{s+3} \rightarrow \text{improper}$$

numerator $\rightarrow s^2$ (2)
denominator $\rightarrow s$ (1)
$2>1$

$$\begin{array}{r|l} & s+4 \\ \hline s+3 & s^2+7s+5 \\ & s^2+3s \\ \hline & 0 \quad 4s+5 \\ & \quad\; 4s+12 \\ \hline & \qquad -7 \end{array}$$

$$\frac{s^2+7s+5}{s+3} = s+4+\frac{-7}{s+3}$$

$$\begin{array}{r|l} & 3 \\ \hline 5 & 17 \\ & 15 \\ \hline & 2 \end{array}$$

$$\frac{17}{5} = 3 + \frac{2}{5}$$

$$f(t) = \delta(t) \longrightarrow F(s) = 1$$

$$f(t) = \frac{d}{dt}\delta(t) \longrightarrow F(s) = s$$

$$f(t) = \frac{d^2}{dt^2}\delta(t) \longrightarrow F(s) = s^2$$

$$F(s) = \frac{s^2+7s+5}{s+3} = s+4+\frac{-7}{s+3}$$

$$f(t) = \frac{d}{dt}\delta(t) + 4\delta(t) - 7e^{-3t}$$

$$F(s) = \frac{s^3+7s^2+9s+7}{s^2+3s+2}$$

$$\begin{array}{r|l} & s+4 \\ \hline s^2+3s+2 & s^3+7s^2+9s+7 \\ - & s^3+3s^2+2s \\ \hline & 4s^2+7s+7 \\ - & 4s^2+12s+8 \\ \hline & -5s-1 \end{array}$$

$$F(s) = \frac{s^3+5s^2+9s+7}{s^2+3s+2} = s+4+\frac{-5s-1}{s^2+3s+2}$$

$$\frac{-5s-1}{s^2+3s+2} = \frac{-5s-1}{(s+2)(s+1)} = \frac{A}{s+2}+\frac{B}{s+1}$$

$$A = \left.\frac{-5s-1}{s+1}\right|_{s=-2} = \frac{-5(-2)-1}{-2+1} = -9$$

$$B = \left.\frac{-5s-1}{s+2}\right|_{s=-1} = \frac{-5(-1)-1}{-1+2} = 4$$

$$F(s) = s+4+\frac{-9}{s+2}+\frac{4}{s+1}$$

$$f(t) = \frac{d}{dt}\delta(t)+4\delta(t)-9e^{-2t}+4e^{-t}$$

$$F(s) = \frac{s^4 + 13s^3 + 66s^2 + 200s + 300}{s^2 + 9s + 20}$$ 352

$$\begin{array}{r|l} & s^2 + 4s + 10 \\ \hline s^2+9s+20 & s^4 + 13s^3 + 66s^2 + 200s + 300 \\ & -\;(s^4 + 9s^3 + 20s^2) \\ \hline & 4s^3 + 46s^2 + 200s + 300 \\ & -\;(4s^3 + 36s^2 + 80s) \\ \hline & 10s^2 + 120s + 300 \\ & -\;(10s^2 + 90s + 200) \\ \hline & 30s + 100 \end{array}$$

$$F(s) = s^2 + 4s + 10 + \frac{30s + 100}{s^2 + 9s + 20}$$

$$\frac{30s+100}{s^2+9s+20} = \frac{30s+100}{(s+4)(s+5)} = \frac{-20}{s+4} + \frac{50}{s+5}$$

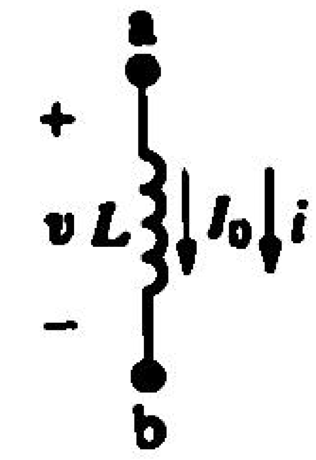
$$F(s) = s^2 + 4s + 10 + \frac{-20}{s+4} + \frac{50}{s+5}$$

$$f(t) = \frac{d^2}{dt^2}\delta(t) + 4\frac{d}{dt}\delta(t) + 10\,\delta(t) - 20e^{-4t} + 50e^{-5t}$$

An Inductor in the s Domain

Figure 13.2 shows an inductor carrying an initial current of I_0 amperes. The time-domain equation that relates the terminal voltage to the terminal current is

$$v = L\frac{di}{dt}. \tag{13.3}$$

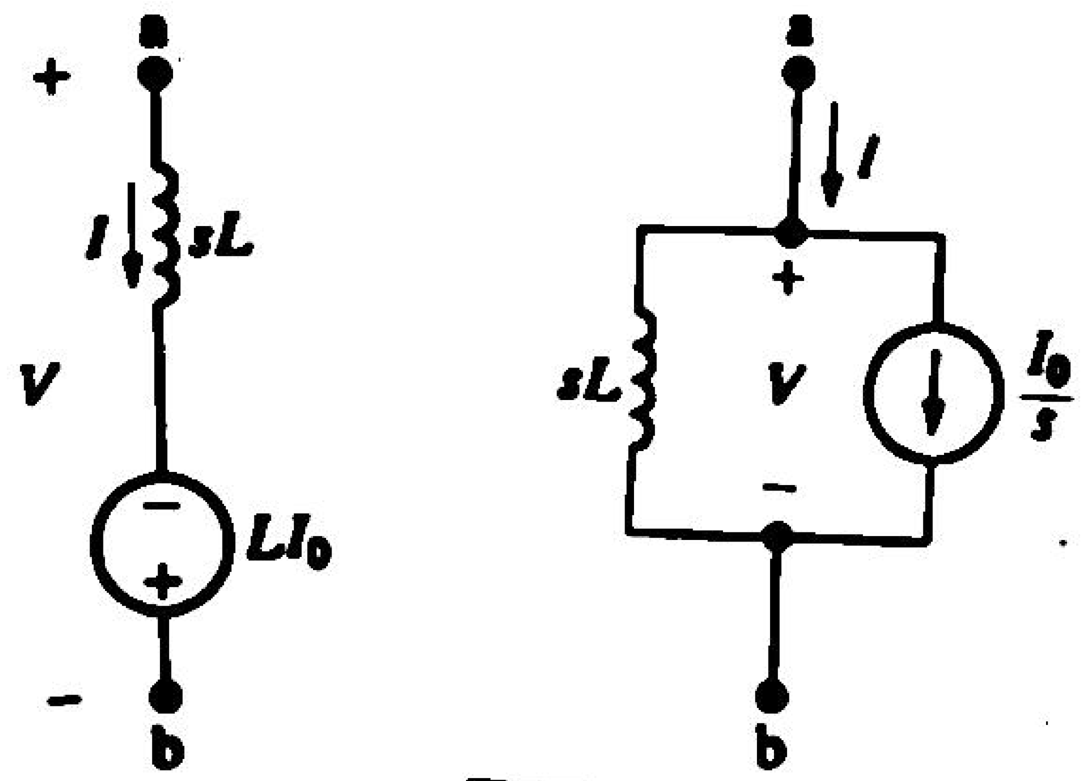


The Laplace transform of Eq. 13.3 gives

$$V = L[sI - i(0^-)] = sLI - LI_0. \tag{13.4}$$

$V + LI_0 = sLI$

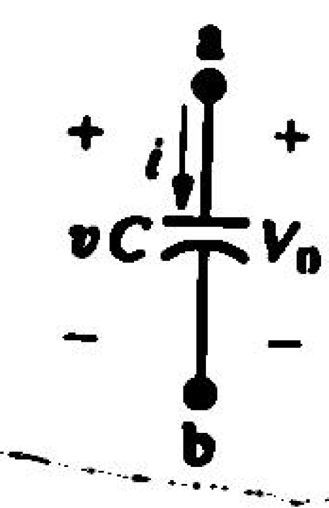
$$I = \frac{V + LI_0}{sL} = \frac{V}{sL} + \frac{I_0}{s}. \tag{13.5}$$



A Capacitor in the s Domain

An initially charged capacitor also has two s-domain equivalent circuits. Figure 13.6 shows a capacitor initially charged to V_0 volts. The terminal current is

$$i = C\frac{dv}{dt}. \tag{13.6}$$



Transforming Eq. 13.6 yields

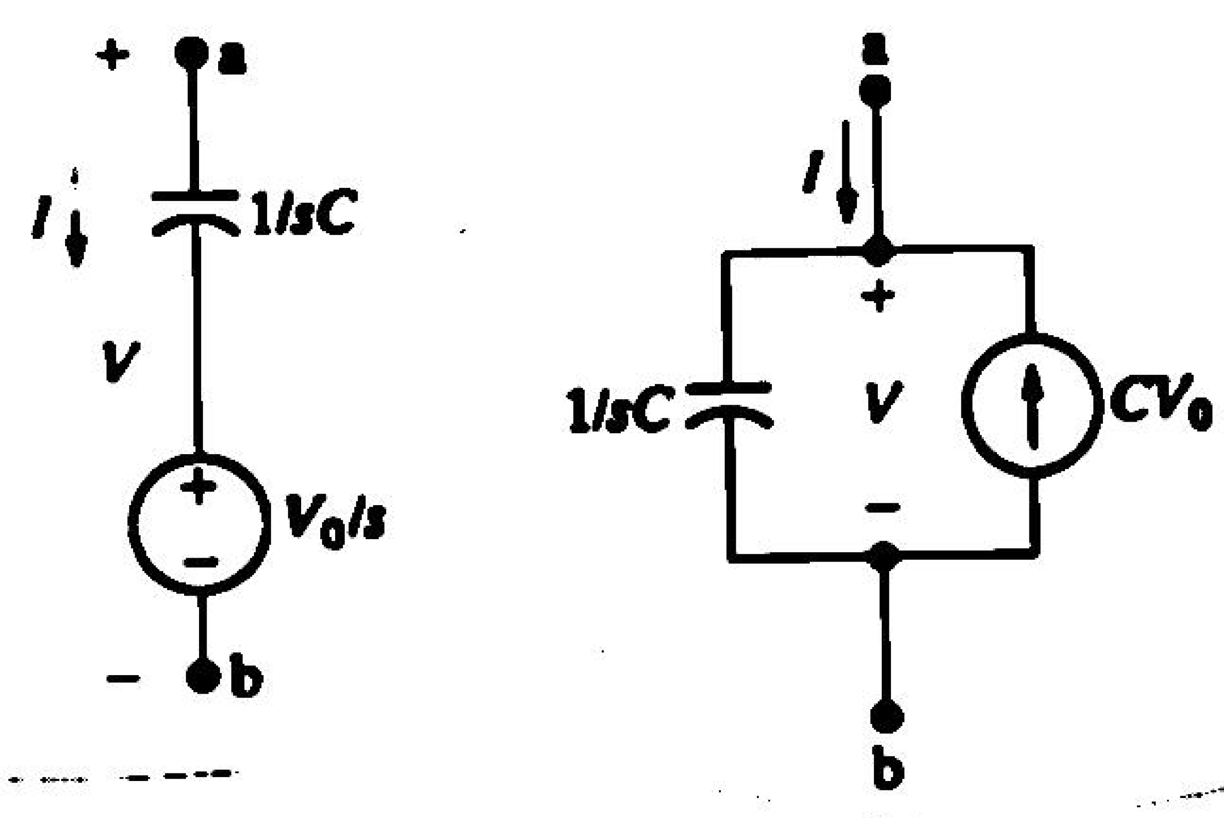
$$I = C[sV - v(0^-)]$$

or

$$I = sCV - CV_0. \tag{13.7}$$

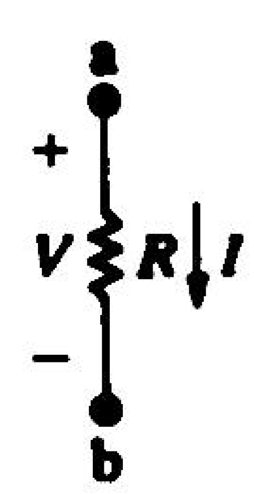
$I + CV_0 = sCV$

$$V = \left(\frac{1}{sC}\right)I + \frac{V_0}{s}. \tag{13.8}$$

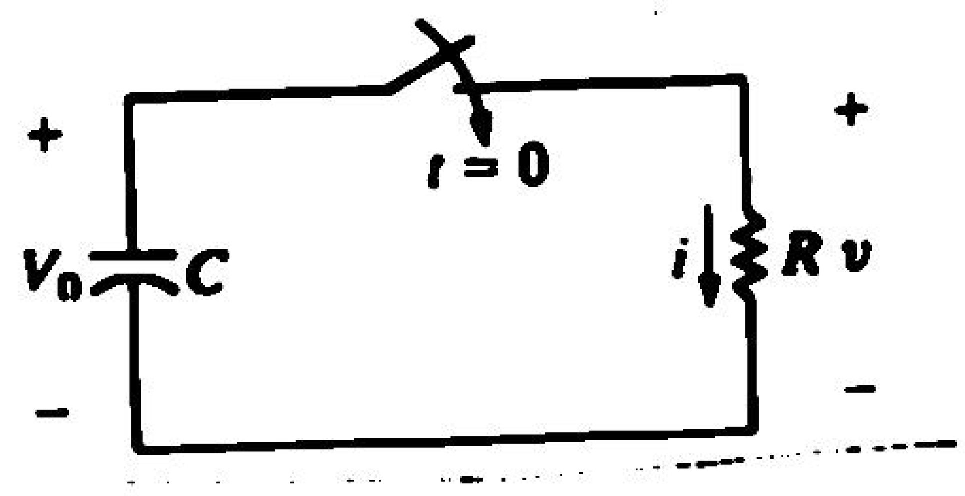


A Resistor in the s Domain

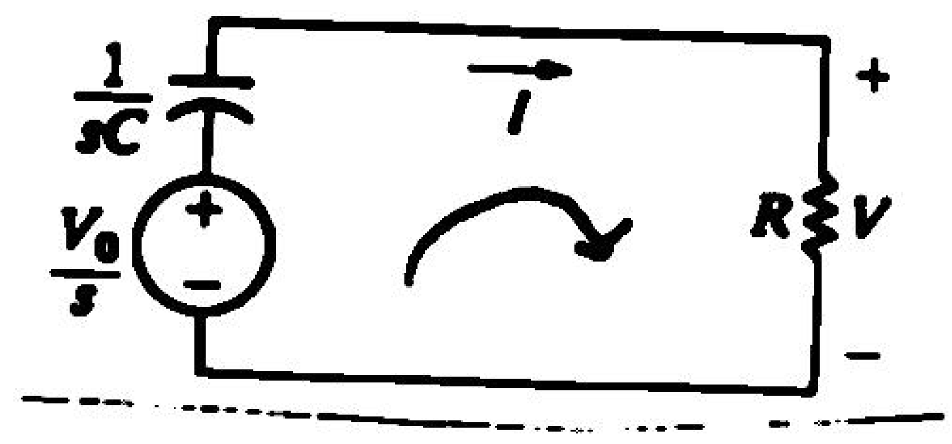
$v = Ri.$ $\qquad V = RI,$



(b)



$\Rightarrow$



$$-\frac{V_0}{s} + \left(\frac{1}{sc}\right) I + RI = 0$$

$$\left[\frac{1}{sc} + R\right] I = \frac{V_0}{s}$$

$$\frac{1 + Rsc}{sc} I = \frac{V_0}{s}$$

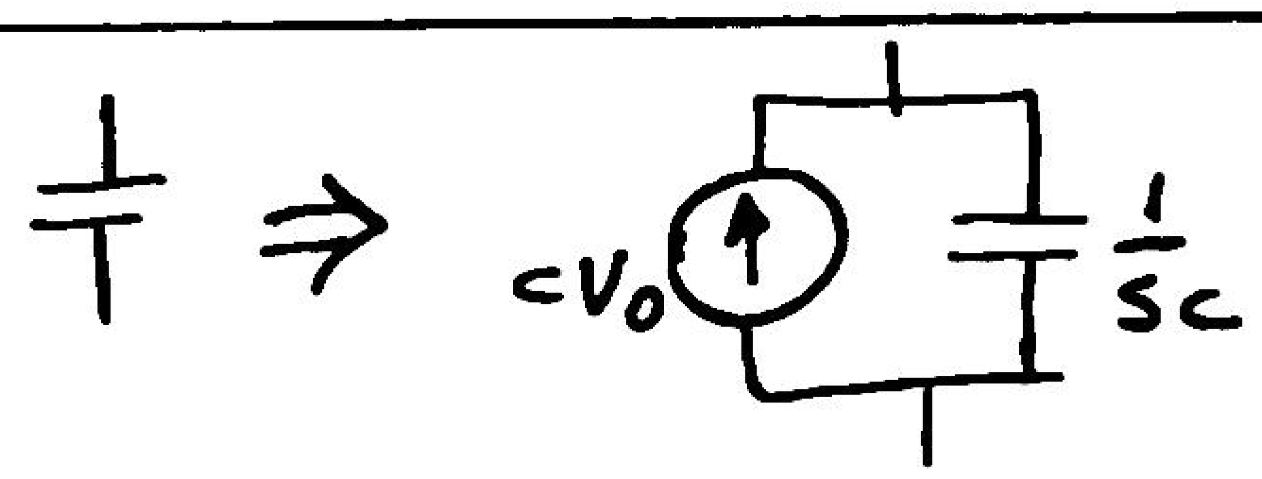
$$I = \frac{V_0}{s} \frac{sc}{1 + Rcs} = \frac{V_0 C}{1 + RCS} = \frac{V_0/R}{s + \frac{1}{RC}}$$

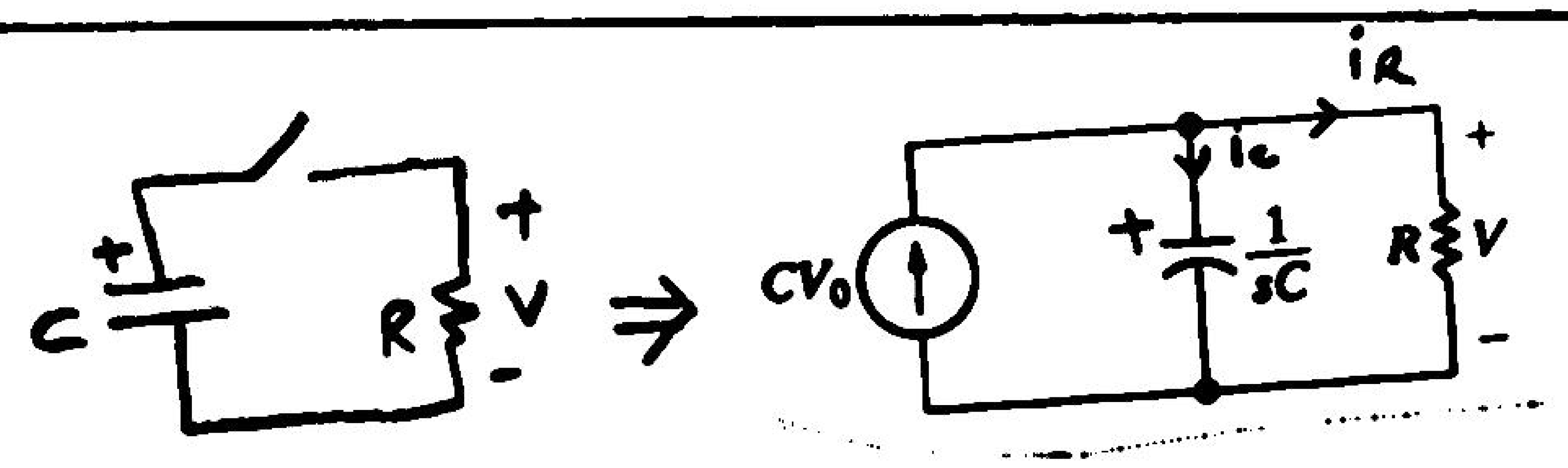
$$f(s) = \frac{a}{s+b} \longrightarrow f(t) = a e^{-bt}$$

$$I(s) = \frac{V_0/R}{s + \frac{1}{RC}} \longrightarrow i(t) = \frac{V_0}{R} e^{-\frac{1}{RC} t}$$

The voltage across the resistor is $V(t) = Ri(t)$

$$V(t) = R\, i(t) = V_0 e^{-\frac{1}{RC} t}$$





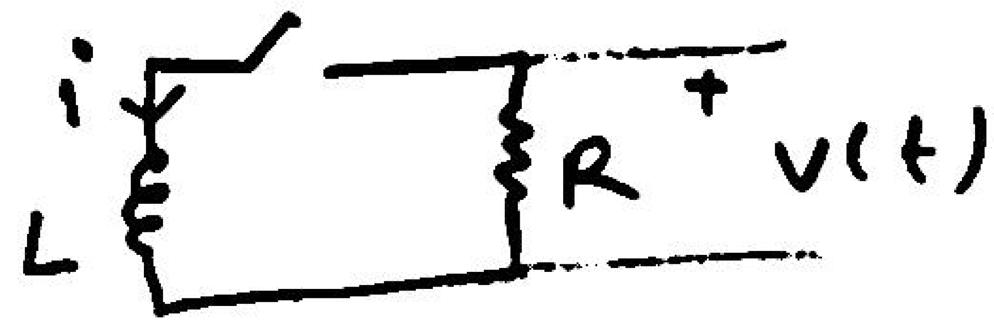
node equation $cV_0 = i_c + i_R$

$$cV_0 = \frac{V}{\frac{1}{sc}} + \frac{V}{R} \Rightarrow cV_0 = scV + \frac{V}{R}$$

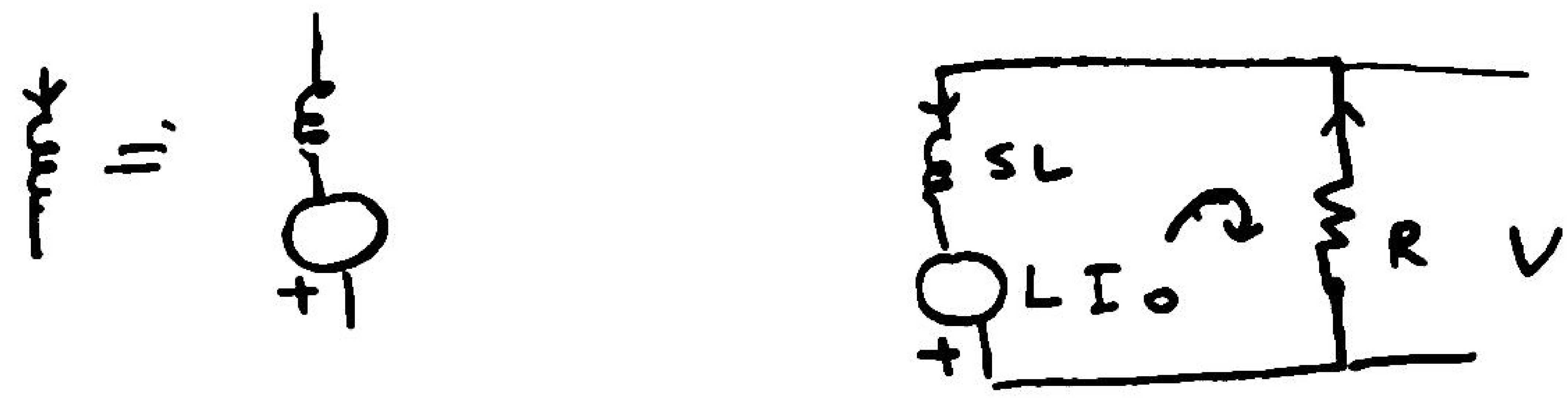
$$cV_0 = \left(sc + \frac{1}{R}\right) V \Rightarrow V = \frac{cV_0}{sc + \frac{1}{R}} = \frac{c V_0/c}{s + \frac{1}{Rc}} = \frac{V_0}{s + \frac{1}{Rc}}$$

$$V(s) = \frac{V_0}{s + \frac{1}{Rc}} \Rightarrow \boxed{v(t) = V_0 e^{-\frac{1}{Rc} t}}$$

Example Problem: find $i(t)$, $v(t)$



Solution



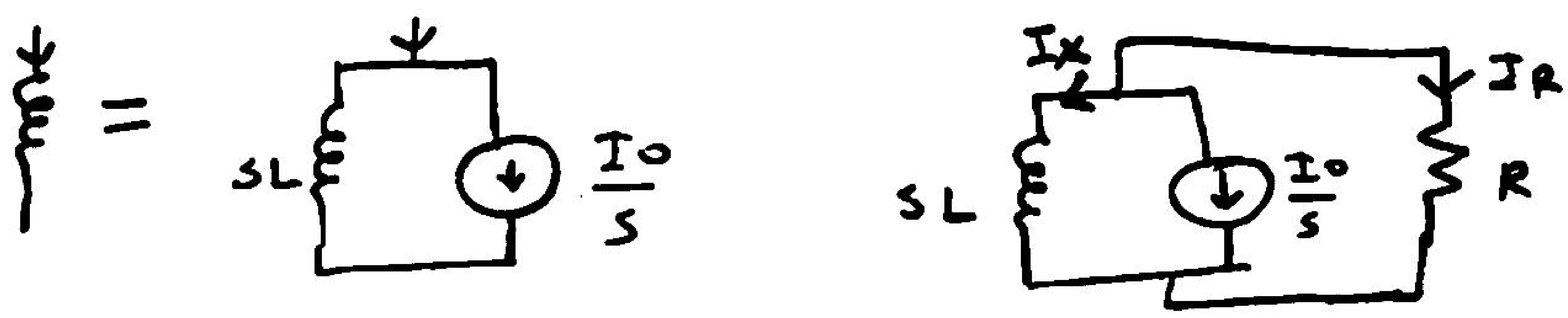
$$+LI_o - sLI - RI = 0$$

$$I(sL+R) = LI_o \Rightarrow I = \frac{LI_o}{sL+R} = \frac{I_o}{s+\frac{R}{L}}$$

$$I(s) = \frac{I_o}{s+\frac{R}{L}} \Rightarrow i(t) = I_o e^{-\frac{R}{L}t}$$

$$v(t) = -R\,i(t) = -RI_o e^{-\frac{R}{L}t}$$

Second method

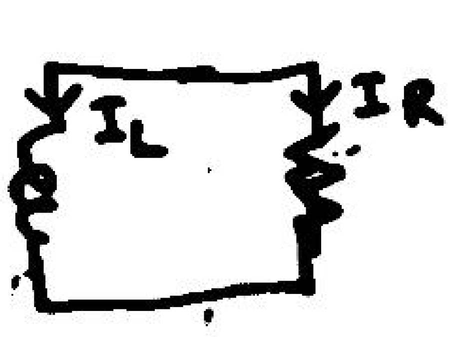


node equation $\quad I_x + \frac{I_o}{s} + I_R = 0$

$$\frac{V}{sL} + \frac{I_o}{s} + \frac{V}{R} = 0$$

$$V\left(\frac{1}{sL} + \frac{1}{R}\right) = -\frac{I_o}{s} \Rightarrow V = \frac{-\frac{I_o}{s}}{\frac{1}{sL}+\frac{1}{R}} = \frac{-I_oL}{1+\frac{sL}{R}} = \frac{-I_oR}{\frac{R}{L}+s}$$

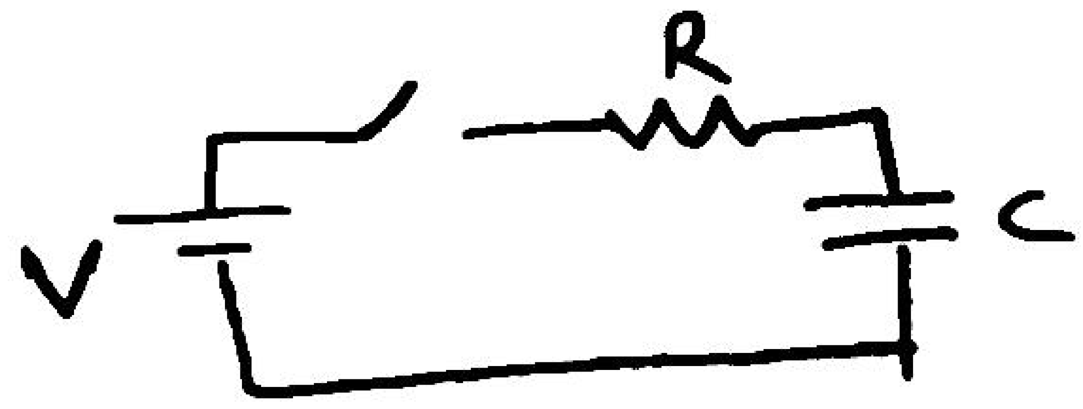
$$V(s) = -\frac{I_oR}{s+\frac{R}{L}} \Rightarrow v(t) = -I_oRe^{-\frac{R}{L}t}$$



$i_R(t) = \frac{v(t)}{R}$, $\quad i_L(t) = -i_R(t) = \frac{-v(t)}{R} \Rightarrow i_L(t) = I_o e^{-\frac{R}{L}t}$

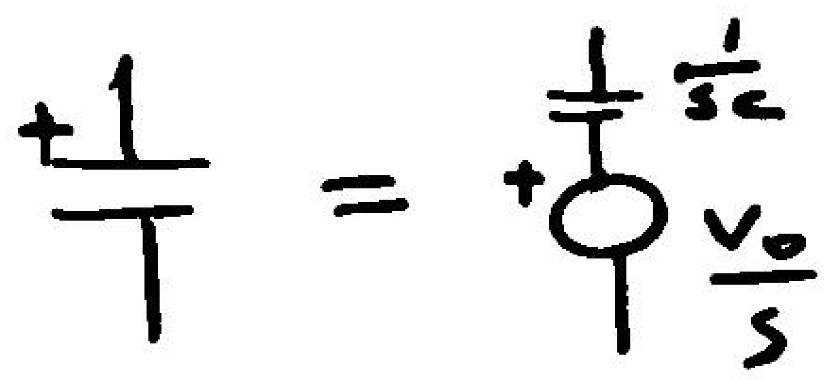
Step Response

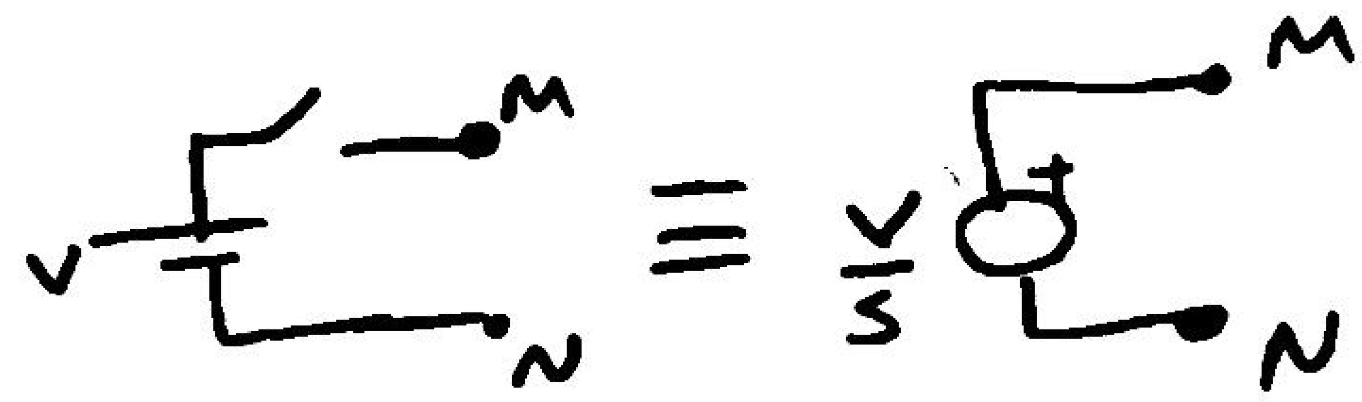
Example problem: calculate $V_c(t)$ if $V_0 = V_c(0) = 20^V$



$R = 2^{\Omega}$, $C = 0.1\,F$, $V = 10^V$

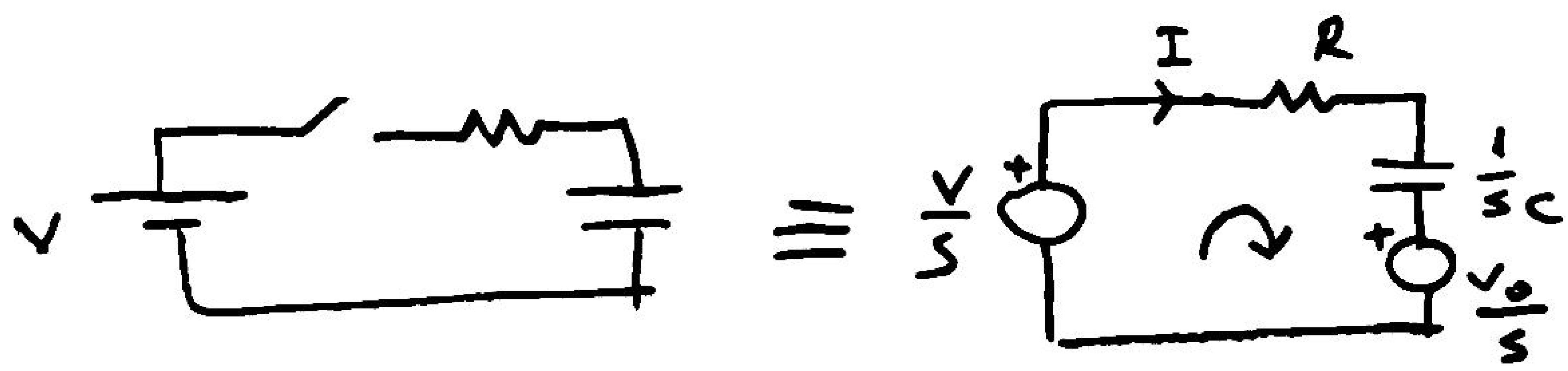
Solution





We represent DC source and switch as a step function

$\Rightarrow \mathcal{L}\{V\,u(t)\} = \frac{V}{s}$



$$-\frac{V}{s} + RI + \frac{1}{sC}I + \frac{V_0}{s} = 0$$

$$I\left(R + \frac{1}{sC}\right) = \frac{V}{s} - \frac{V_0}{s}$$

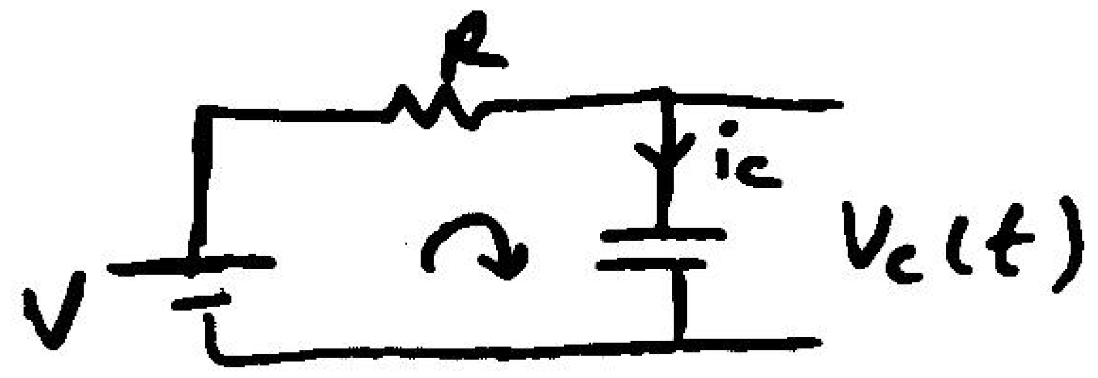
$$I = \frac{\frac{V}{s} - \frac{V_0}{s}}{R + \frac{1}{sC}} = \frac{V - V_0}{sR + \frac{1}{C}} = \frac{(V - V_0)/R}{s + \frac{1}{RC}}$$

$$I(s) = \frac{(V - V_0)/R}{s + \frac{1}{RC}} \Rightarrow i(t) = \frac{V - V_0}{R} e^{-\frac{1}{RC}t}$$

$V = 10$, $R = 2$ $C = 0.1$ $V_0 = 20$ $\Rightarrow i(t) = \frac{10 - 20}{2} e^{-\frac{1}{0.2}t} = -5\,e^{-5t}$

Now find $V_c(t)$. $\quad i_c(t) = \frac{V-V_o}{R} e^{-\frac{1}{Rc}t}$

method 1



$$-V + R i_c + V_c = 0$$

$$V_c = V - R i_c$$

$$= V - R \frac{V-V_o}{R} e^{-\frac{1}{Rc}t}$$

$$= V - (V - V_o) e^{-\frac{1}{Rc}t}$$

$$V_c(t) = 10 - (10-20) e^{-5t}$$

$$= 10 + 10 e^{-5t}$$

method 2

$$I_c(t) = c \frac{dV_c}{dt} \Rightarrow V_c(t) = \frac{1}{c} \int_0^t i_c(\alpha) d\alpha + V_c(0)$$

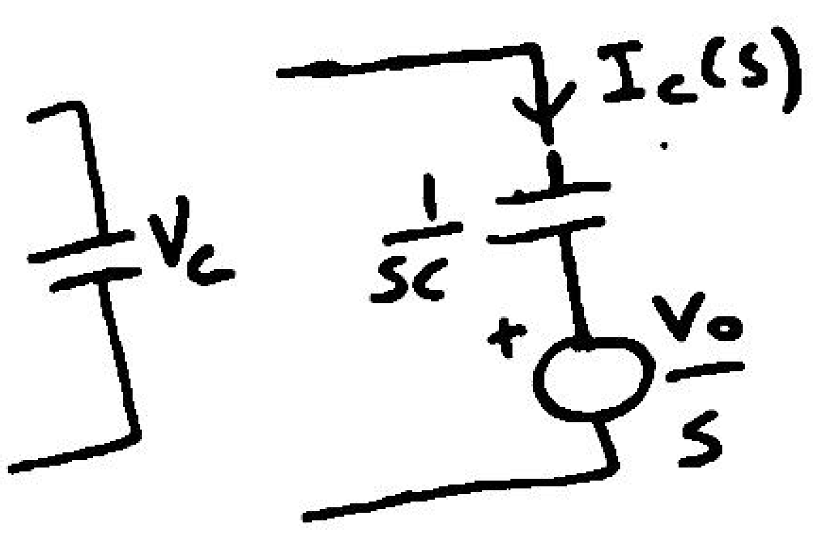
$$i_c(t) = -5 e^{-5t} \qquad I_c(\alpha) = -5 e^{-5\alpha}$$

$$V_c(t) = \frac{1}{0.1} \int_0^t -5 e^{-5\alpha} d\alpha + 20 = \frac{-5}{0.1} \int_0^t e^{-5\alpha} d\alpha + 20$$

$$= \frac{-5}{0.1} \frac{1}{-5} e^{-5\alpha} \Big|_0^t + 20 = 10 (e^{-5t} - e^0) + 20 = 10 + 10 e^{-5t}$$

method 3

$$V_c(s) = \frac{1}{sc} I_c(s) + \frac{V_o}{s} = \frac{1}{sc} \frac{(V-V_o)/R}{s + \frac{1}{Rc}} + \frac{V_o}{s}$$



$$= \frac{1}{0.1s} \frac{(10-20)/2}{s + \frac{1}{2 \times 0.1}} + \frac{20}{s} = \frac{-50}{s(s+5)} + \frac{20}{s}$$

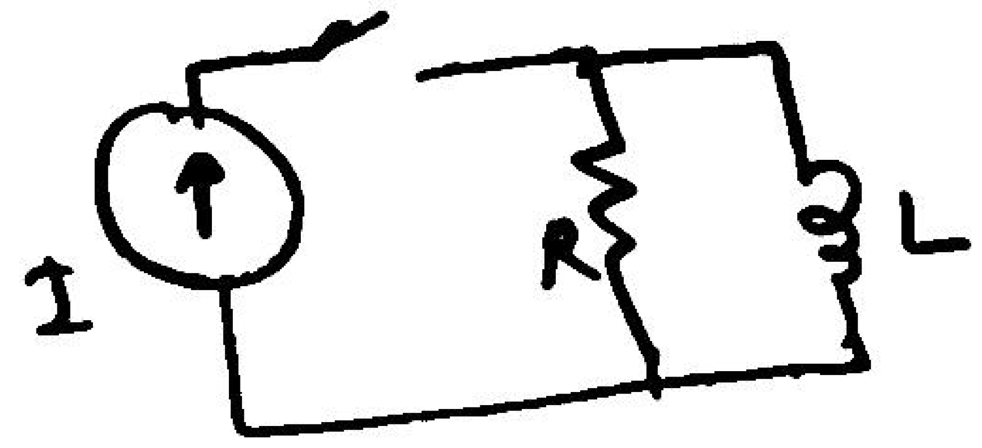
$$\frac{1}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5} \Longrightarrow A = 0.2 \qquad B = -0.2$$

$$V_c(s) = -50 \left(\frac{0.2}{s} + \frac{-0.2}{s+5} \right) + \frac{20}{s} = -\frac{10}{s} + \frac{10}{s+5} + \frac{20}{s}$$

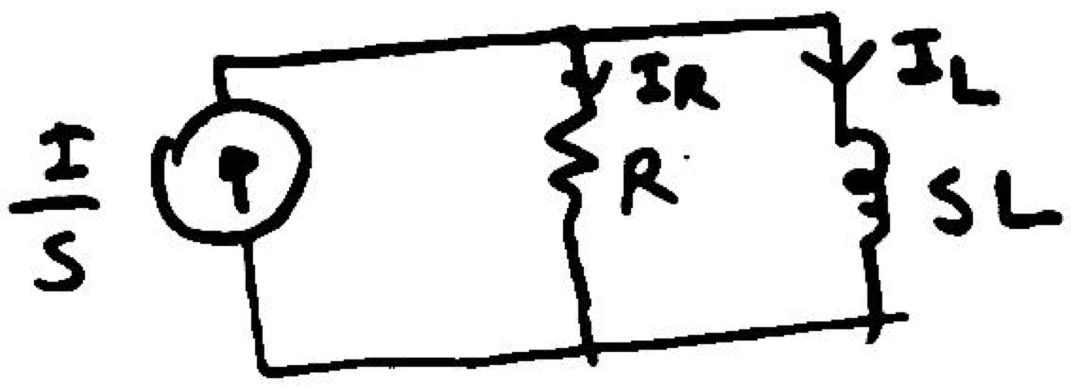
$$= \frac{10}{s} + \frac{10}{s+5} \Rightarrow V_c(t) = 10 + 10 e^{-5t}$$

Example 74: Calculate $V_L(t)$, $I_L(t)$, $I_L(0)=0$

$R=2$, $L=1$, $I=10\,A$



Solution



$$\frac{I}{s}=I_R+I_L \longrightarrow \frac{I}{s}=\frac{V}{R}+\frac{V}{SL} \longrightarrow \frac{I}{s}=\left(\frac{1}{R}+\frac{1}{SL}\right)V$$

$$V=\frac{\frac{I}{s}}{\frac{1}{R}+\frac{1}{SL}}=\frac{I}{\frac{s}{R}+\frac{1}{L}}=\frac{RI}{s+\frac{R}{L}}$$

$$V(s)=\frac{RI}{s+\frac{R}{L}} \longrightarrow V(t)=RI\,e^{-\frac{R}{L}t}$$

$$V(t)=2\times 10\,e^{-\frac{2}{1}t}=20\,e^{-2t}$$

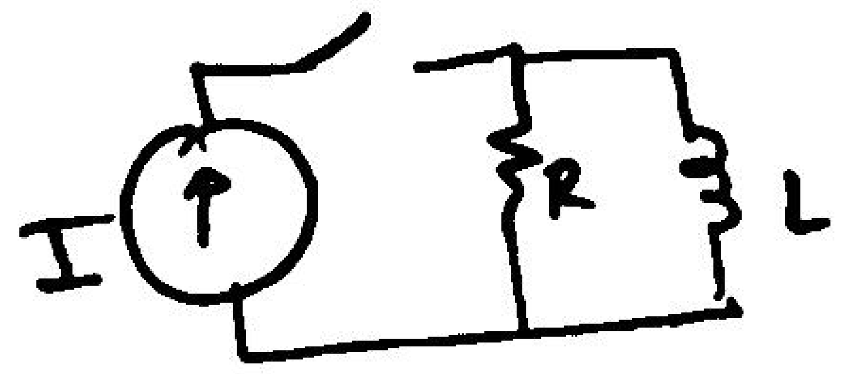
$$I_L(S)=\frac{V(s)}{SL}=\frac{1}{SL}\left(\frac{RI}{s+\frac{R}{L}}\right)=\frac{1}{s\times 1}\;\frac{2\times 10}{(s+\frac{2}{1})}=\frac{20}{s(s+2)}$$

$$=\frac{A}{s}+\frac{B}{s+2} \qquad A=10 \qquad B=-10$$

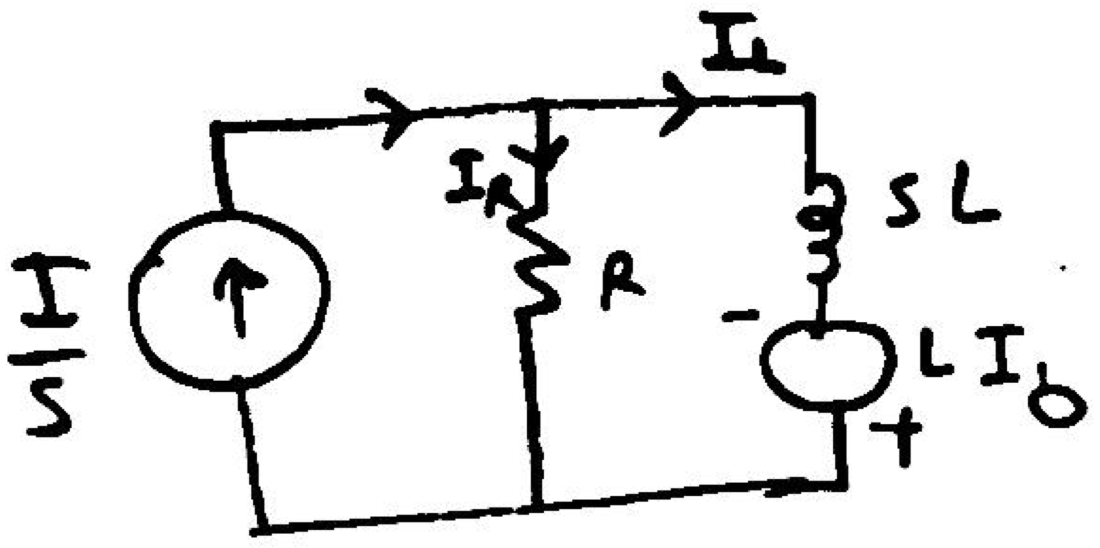
$$I_L(s)=\frac{10}{s}-\frac{10}{s+2} \longrightarrow I_L(t)=10\,u(t)-10\,e^{-2t}$$

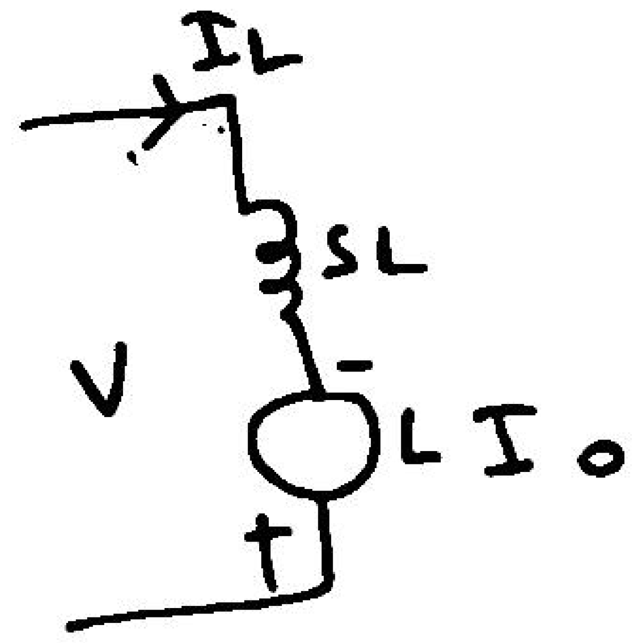
Example 75 Calculate $V_L(t)$, $I_L(t)$ $\quad I_L(0) = I_0 = 20\,A$

$R = 2 \quad L = 1 \quad I = 10\,A$



Solution





$$V = SL\,I_L - L\,I_0$$

$$I_L = \frac{V + LI_0}{SL}$$

$$\frac{I}{s} = I_R + I_L$$

$$\frac{I}{s} = \frac{V}{R} + \frac{V + LI_0}{SL} \rightarrow \frac{I}{s} - \frac{LI_0}{SL} = \left(\frac{1}{R} + \frac{1}{SL}\right)V$$

$$V = \frac{\frac{I}{s} - \frac{I_0}{s}}{\frac{1}{R} + \frac{1}{SL}} = \frac{I - I_0}{\frac{S}{R} + \frac{1}{L}} = \frac{R(I - I_0}{S + \frac{R}{L}} = \frac{2(10 - 20)}{S + \frac{2}{1}}$$

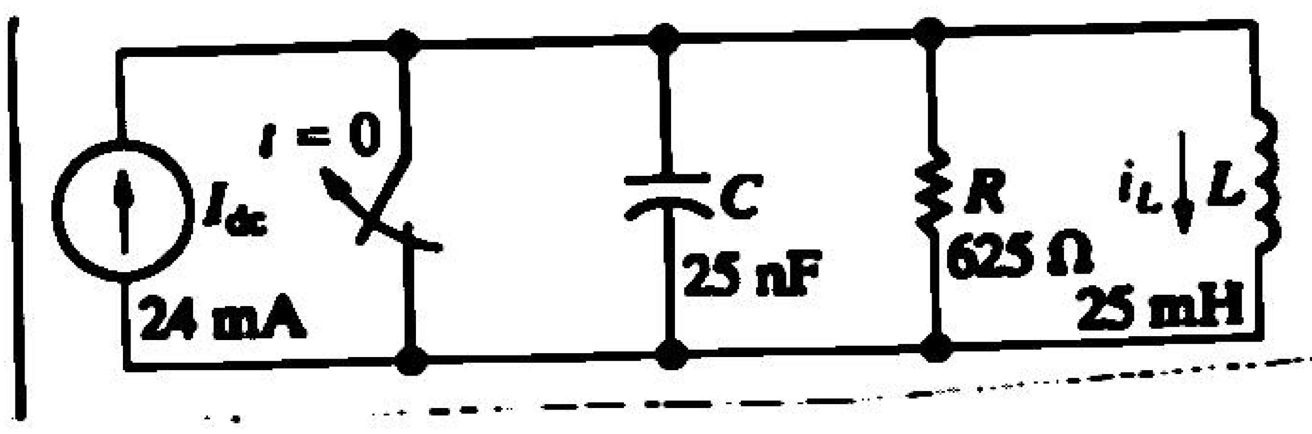
$$V = \frac{-20}{s+2} \rightarrow v(t) = -20\,e^{-2t}$$

$$I_L(s) = \frac{V + LI_0}{SL} = \frac{-\frac{20}{s+2} + LI_0}{SL} = \frac{-\frac{20}{s+2} + 1 \times 20}{1 \times S} = \frac{20}{S} - \frac{20}{s(s+2)}$$

$$\frac{20}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2} \qquad A = 10 \qquad B = -10$$

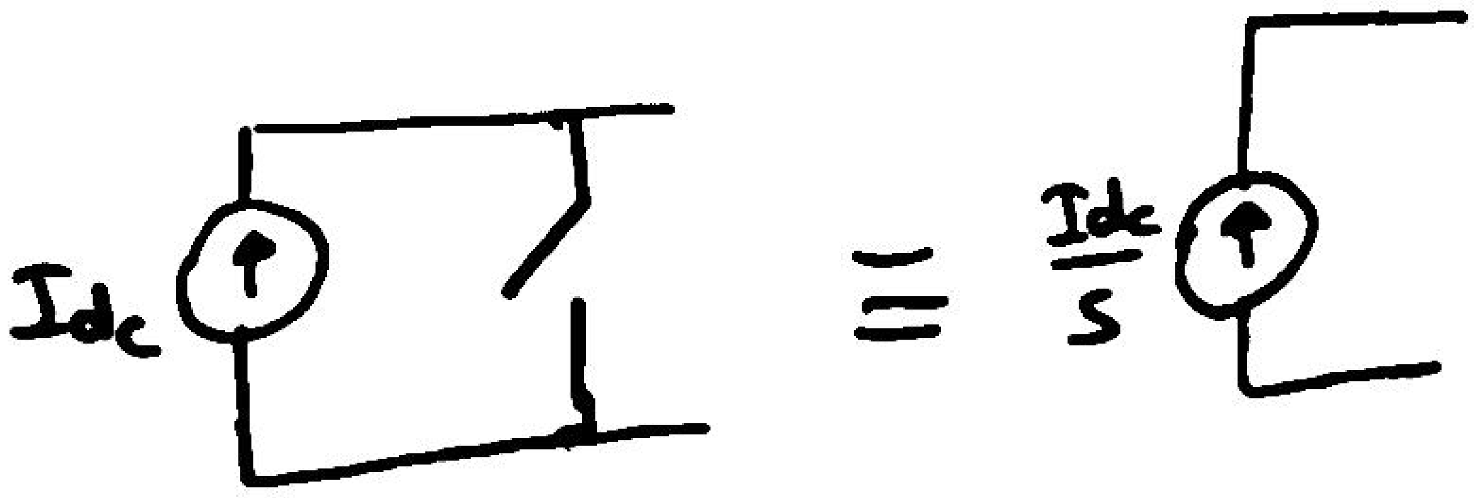
$$I_L(s) = \frac{20}{s} - \frac{20}{s(s+2)} = \frac{20}{s} - \left(\frac{10}{s} - \frac{10}{s+2}\right) = \frac{10}{s} + \frac{10}{s+2}$$

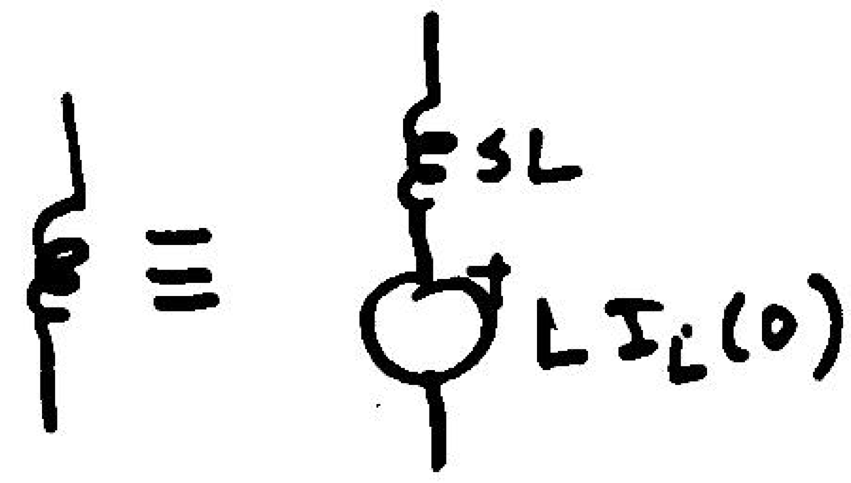
$$I_L(t) = 10\,u(t) + 10\,e^{-2t}$$

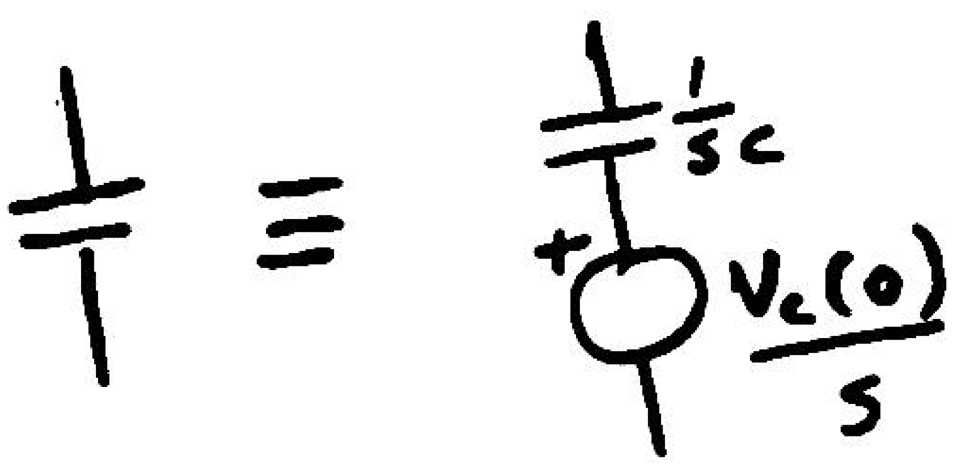


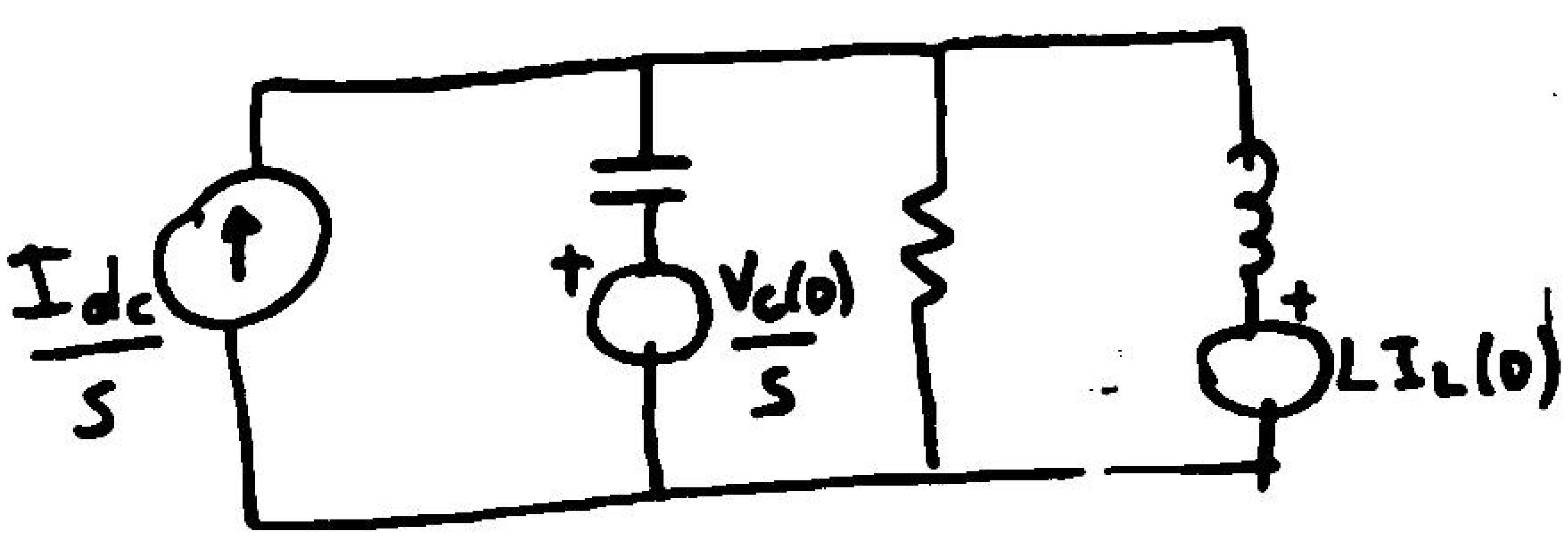
Calculate currents and Voltage if $I_L(0) = 0$ $V_c(0) = 0$

solution

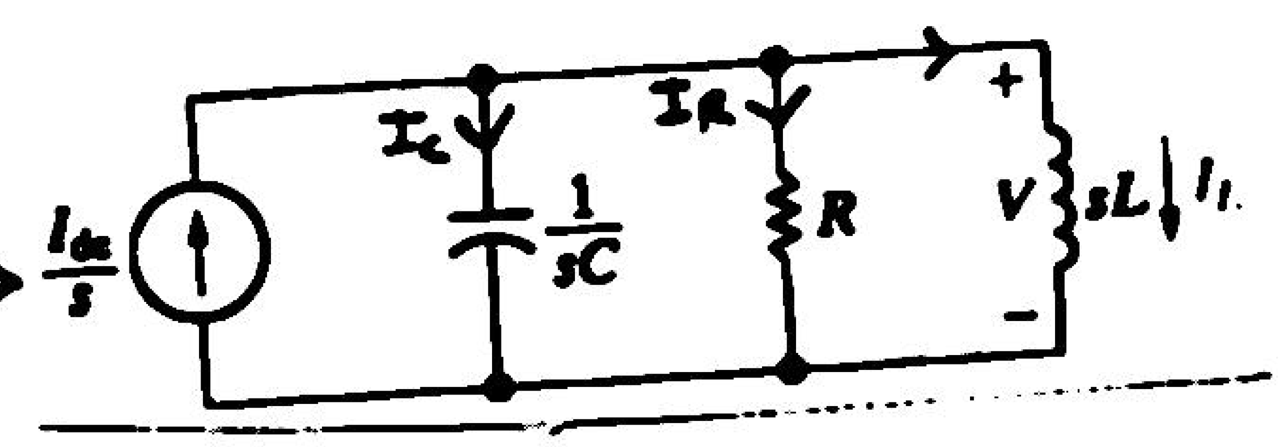








$I_L(0) = 0$
$V_c(0) = 0$ $\Rightarrow$



$$\frac{I_{dc}}{s} = I_c + I_R + I_L$$

$$\frac{I_{dc}}{s} = \frac{V}{\frac{1}{sc}} + \frac{V}{R} + \frac{V}{sL} \Rightarrow \frac{I_{dc}}{s} = \left(sc + \frac{1}{R} + \frac{1}{sL}\right) V$$

$$V = \frac{I_{dc}}{s} \frac{1}{sc + \frac{1}{R} + \frac{1}{sL}} = \frac{I_{dc}}{s} \frac{sL}{s^2 Lc + s\frac{L}{R} + 1} = \frac{I_{dc}/c}{s^2 + \frac{1}{Rc} s + \frac{1}{Lc}}$$

$$V = \frac{24 \times 10^{-3} / 25\,10^{-9}}{s^2 + \frac{1}{625 \times 25 \times 10^{-9}} s + \frac{1}{25 \times 10^{-3}\, 25 \times 10^{-9}}} = \frac{960000}{s^2 + 64000s + 16 \times 10^8}$$

$$V(s) = \frac{960000}{s^2 + 64000s + 16\times10^8}$$

$$s^2 + 64000s + 16\times10^8 = 0 \quad \begin{matrix} \nearrow s_1 = -32000 + 24000j \\ \searrow s_2 = -32000 - 24000j \end{matrix}$$

$$s^2 + 64000s + 16\times10^8 = (s+32000)^2 + 24000^2$$

$$V(s) = \frac{960000}{24000} \cdot \frac{24000}{(s+32000)^2 + 24000^2}$$

$$\mathcal{L}^{-1}\left\{ \frac{b}{(s+a)^2 + b^2} \right\} = e^{-at} \sin bt$$

$$v(t) = \frac{960000}{24000} \cdot e^{-32000t} \sin 24000t$$

Now calculate $i_L(t)$ and $i_c(t)$

$$sL \quad I_L = \frac{V(s)}{sL} = \frac{V(s)}{25\times10^{-3}s} = \frac{960000}{25\times10^{-3}\,s} \quad \frac{1}{s^2 + 64000s + 16\times10^8}$$

$$I_L(s) = \frac{3{,}8400000}{s(s^2 + 64000s + 16\times10^8)} = \frac{A}{s} + \frac{B}{s_1} + \frac{C}{s_2}$$

Here $s_1 = -32000 + 24000j \qquad s_2 = -32000 - 24000j$

$$A = s \cdot \left.\frac{3840000}{s(s^2 + 64000s + 16\times10^8)}\right|_{s=0} = \frac{3840000}{16\times10^8} = 0{,}024$$

$$B = (s - s_1) \left.\frac{3840000}{s(s-s_1)(s-s_2)}\right|_{s=s_1} = \frac{3840000}{s_1(s_1 - s_2)}$$

$$\frac{3840000}{(-32000+24000j)\left[-32000+24000j - (-32000-24000j)\right]} = -0{,}012 + 0{,}016j$$

$$C = B^* = (-0.012 + 0.016j)^* = -0.012 - 0.016j$$

$$I_L(s) = \frac{0.024}{s} + \frac{-0.012 + 0.016j}{s-(-32000+24000j)} + \frac{-0.012-0.016j}{s-(-32000-24000j)}$$

$$\mathcal{L}^{-1}\left\{ \frac{a+bi}{s-(x+iy)} + \frac{a-bi}{s-(x-iy)} \right\} = e^{xt}\left[2a \cos yt - 2b \sin yt\right]$$

here $a = -0.012$ $b = 0.016$ $x = -32000$ $y = 24000$

$$i_L(t) = 0.024\, u(t) + e^{-32000t}\big(2\times(-0.012) \cos 24000\, t - 2\times(0.016 \sin 24000 t\big)$$

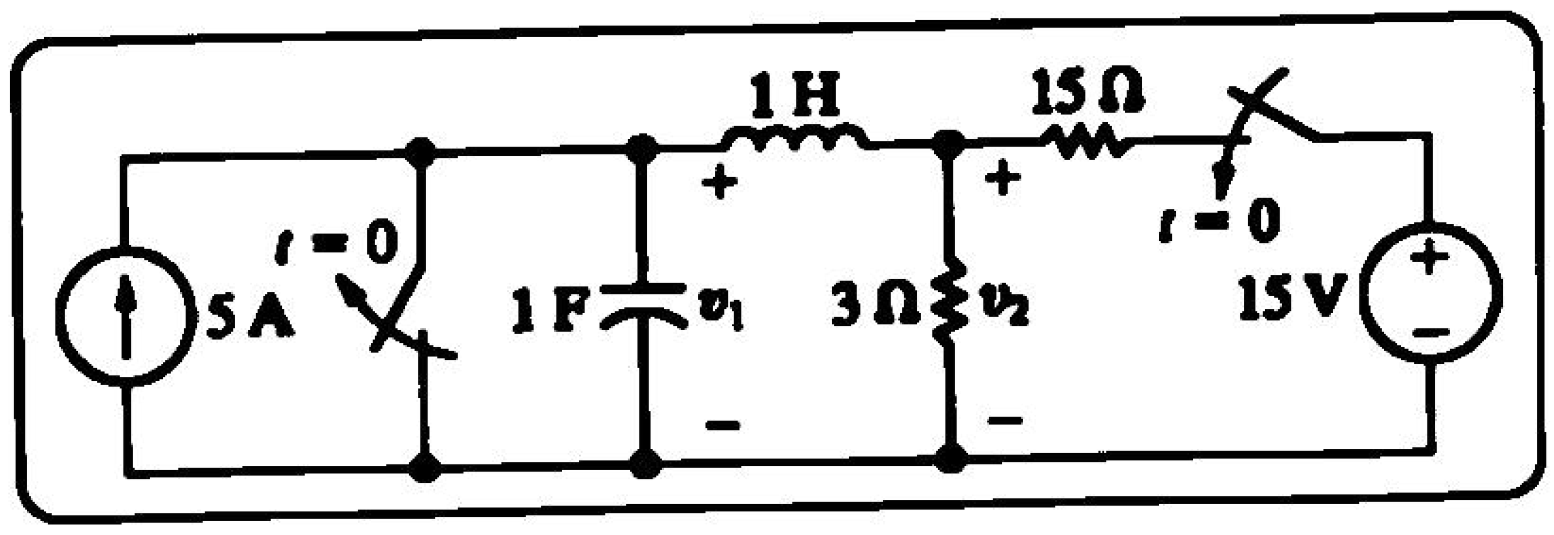
$$i_L(t) = 0.024\, u(t) + e^{-32000t}\left(-0.024 \cos 24000t - 0.032 \sin 24000t\right)$$

$$i_c(t) = i_{dc} - \left[i_R(t) + i_L(t)\right]$$

$$i_R(t) = \frac{v(t)}{R} \qquad i_{dc} = 24$$

replace $v(t)$, i_{dc}, $i_L(t)$, $i_R(t)$ and calculate $i_c(t)$

13.5 The dc current and voltage sources are applied simultaneously to the circuit shown. No energy is stored in the circuit at the instant of application.



ek-1

a) Derive the s-domain expressions for V_1 and V_2.

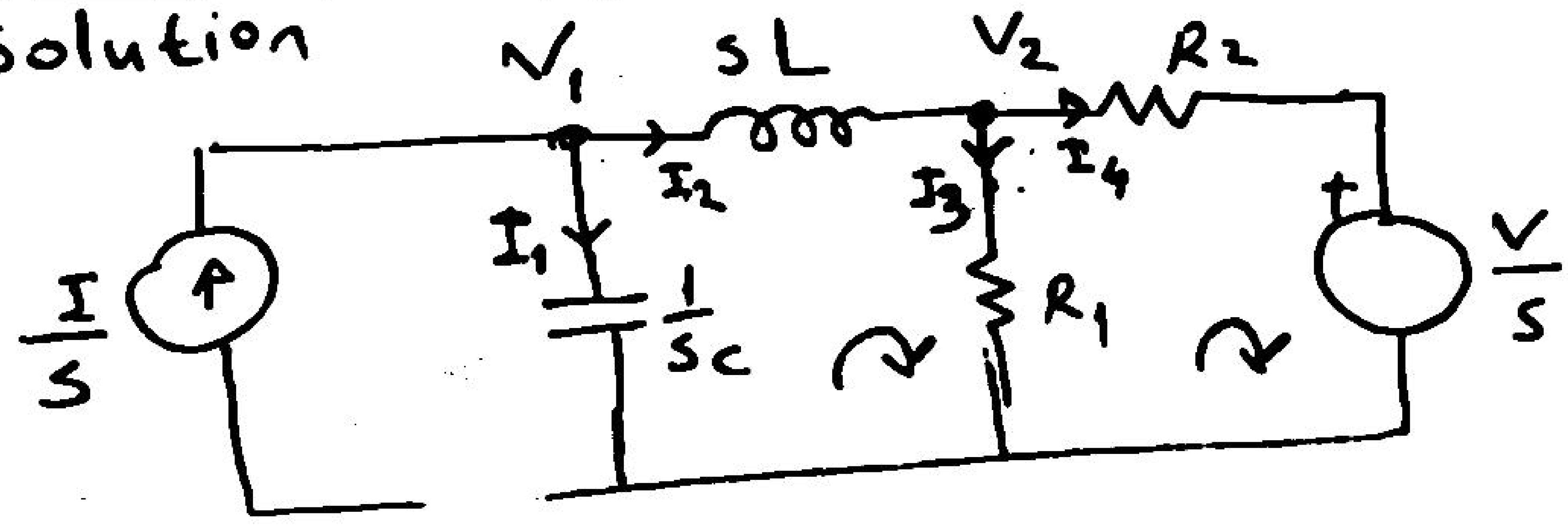
b) For $t > 0$, derive the time-domain expressions for v_1 and v_2.

c) Calculate $v_1(0^+)$ and $v_2(0^+)$.

d) Compute the steady-state values of v_1 and v_2.

ANSWER: (a) $V_1 = [5(s+3)]/[s(s+0.5)(s+2)]$, $V_2 = [2.5(s^2+6)]/[s(s+0.5)(s+2)]$; (b) $v_1 = (15 - \frac{50}{3}e^{-0.5t} + \frac{5}{3}e^{-2t})u(t)$ V, $v_2 = (15 - \frac{125}{6}e^{-0.5t} + \frac{25}{3}e^{-2t})u(t)$ V; (c) $v_1(0^+) = 0$, $v_2(0^+) = 2.5$ V; (d) $v_1 = v_2 = 15$ V.

Solution



$I = 5 \quad V = 15$

$c = 1 \quad L = 1 \quad R_1 = 3$

$R_2 = 15$

node equations $\quad \frac{I}{s} = I_1 + I_2 \quad (1)$

$I_2 = I_3 + I_4 \quad (2)$

loop equations $\quad -V_1 + sL\,I_2 + V_2 = 0 \quad (3)$

$-V_2 + R_2 I_4 + \frac{V}{s} = 0 \quad (4)$

We try to solve V_1 and V_2

$(3) \rightarrow \quad I_2 = \frac{V_1 - V_2}{sL} \quad (5)$

$(4) \rightarrow \quad I_4 = \frac{V_2 - \frac{V}{s}}{R_2} \quad (6)$

$(1),(5) \rightarrow \quad \frac{I}{s} = \frac{V_1}{\frac{1}{sc}} + \frac{V_1 - V_2}{sL} \qquad \frac{I}{s} = sc\,V_1 + \frac{V_1}{sL} - \frac{V_2}{sL}$

$\frac{I}{s} = \underbrace{\left(sc + \frac{1}{sL}\right)}_{A} V_1 - \frac{V_2}{sL} \Rightarrow \frac{I}{s} = A V_1 - B V_2 \quad (7)$

$B = \frac{1}{sL}$

ek-2
[illegible]

$$(2) \rightarrow \quad \underset{\downarrow}{I_2} = \underset{\downarrow}{I_3} + I_4$$

$$(5),(6) \rightarrow \quad \frac{V_1 - V_2}{sL} = \frac{V_2}{R_1} + \frac{V_2 - \frac{V}{s}}{R_2} \rightarrow \frac{V_1}{sL} - \frac{V_2}{sL} = \frac{V_2}{R_1} + \frac{V_2}{R_2} - \frac{V}{sR_2}$$

$$\left(\begin{array}{l} B = \frac{1}{sL} \\ \\ D = \frac{1}{sR_2} \end{array} \right) \qquad BV_1 - BV_2 = \frac{V_2}{R_1} + \frac{V_2}{R_2} - DV$$

$$BV_1 = \underbrace{\left(B + \frac{1}{R_1} + \frac{1}{R_2} \right)}_{E} V_2 + DV$$

$$BV_1 = EV_2 + DV \qquad (8)$$

$$V_1 = \frac{E}{B} V_2 + \frac{D}{B} V \qquad (9)$$

$$(7) \rightarrow \quad \frac{I}{s} = AV_1 - BV_2$$

$$\frac{I}{s} = A\left(\frac{E}{B} V_2 + \frac{D}{B} V \right) - BV_2$$

$$\frac{I}{s} - \frac{AD}{B} V = \left(\frac{AE}{B} - B \right) V_2$$

$$\frac{\frac{I}{s} - \frac{AD}{B} V}{\frac{AE}{B} - B} = V_2 \quad \Rightarrow \quad V_2 = \frac{IB - sADV}{AEs - B^2}$$

$$V_2 = \frac{I \frac{1}{sL} - s\left(sC + \frac{1}{sL} \right) \frac{1}{sR_2} V}{\left(sC + \frac{1}{sL} \right)\left(\frac{1}{sL} + \frac{1}{R_1} + \frac{1}{R_2} \right) s - \frac{1}{(sL)^2}}$$

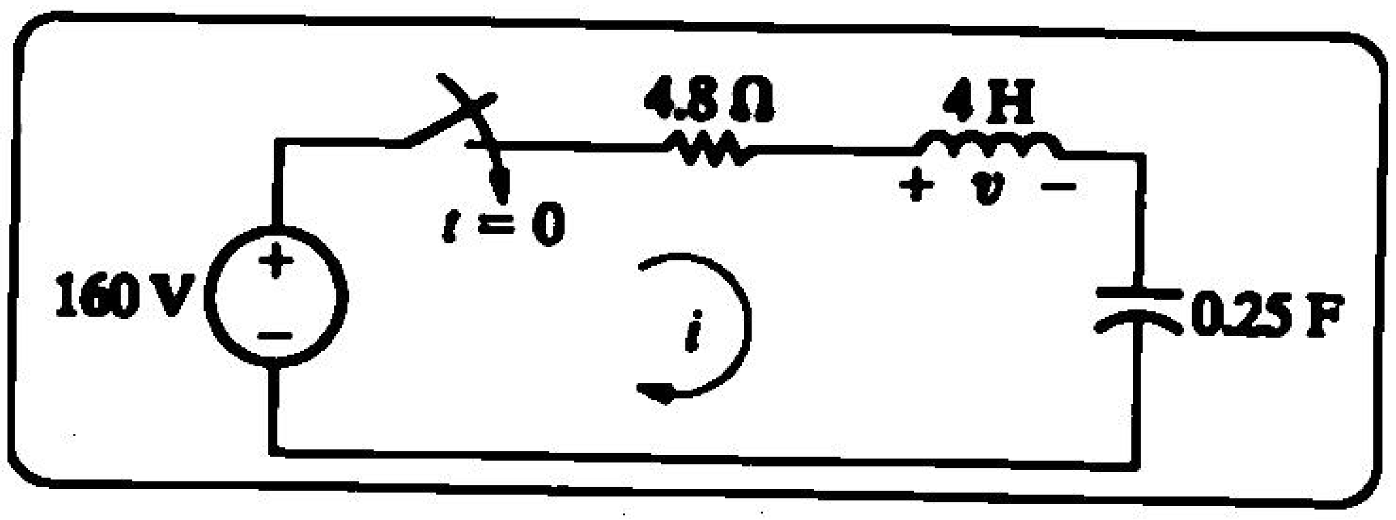
13.4 The energy stored in the circuit shown is zero at the time when the switch is closed.

a) Find the s-domain expression for I.

b) Find the time-domain expression for i when $t > 0$.

c) Find the s-domain expression for V.

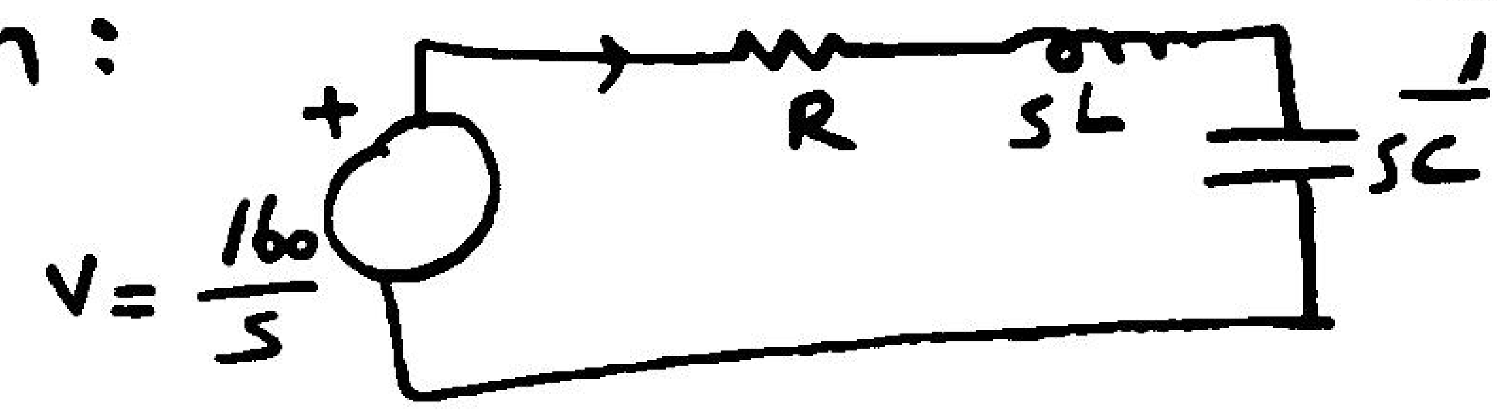
d) Find the time-domain expression for v when $t > 0$.



ANSWER: (a) $I = 40/(s^2 + 1.2s + 1)$;
(b) $i = (50e^{-0.6t}\sin 0.8t)u(t)$ A;
(c) $V = 160s/(s^2 + 1.2s + 1)$;
(d) $v = [200e^{-0.6t}\cos(0.8t + 36.87°)]u(t)$ V.

NOTE ◆ *Also try Chapter Problems 13.15 and 13.16.*

solution:



$$I(s) = \frac{V}{R + sL + \frac{1}{sC}} = \frac{sCV}{sCR + s^2LC + 1} = \frac{sC\frac{160}{s}}{LCs^2 + RCs + 1}$$

$$= \frac{160\,C/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{160/4}{s^2 + \frac{4.8}{4}s + \frac{1}{4 \times 0.25}}$$

$$= \frac{40}{s^2 + 1.2s + 1}$$

$$V(s) = sL\,I(s) = s \times 4\,\frac{40}{s^2 + 1.2s + 1} = \frac{160s}{s^2 + 1.2s + 1}$$

$$s^2 + 1.2s + 1 = 0 \qquad s_1 = \frac{-1.2 \mp \sqrt{1.2^2 - 4}}{2} = -0.6 \mp 0.8j$$

$$s^2 + 1.2s + 1 = (s + 0.6)^2 + 0.8^2$$

$$I(s) = \frac{40}{(s + 0.6)^2 + 0.8^2} = \frac{40}{0.8}\,\frac{0.8}{(s + 0.6)^2 + 0.8^2} \rightarrow i(t) = 50e^{-0.6t}\sin 0.8t$$

$$V(s) = \frac{160s}{(s + 0.6)^2 + 0.8^2} = 160\left(\frac{s + 0.6}{(s + 0.6)^2 + 0.8^2} - \frac{0.6}{(s + 0.6)^2 + 0.8^2}\right).$$

$$V(s) = 160\left(\frac{s+0.6}{(s+0.6)^2+0.8^2} - \frac{0.6}{0.8}\,\frac{0.8}{(s+0.6)^2+0.8^2}\right)$$

$$v(t) = 160\left[e^{-0.6t}\cos 0.8t - 0.75\,e^{-0.6t}\sin 0.8t\right]$$

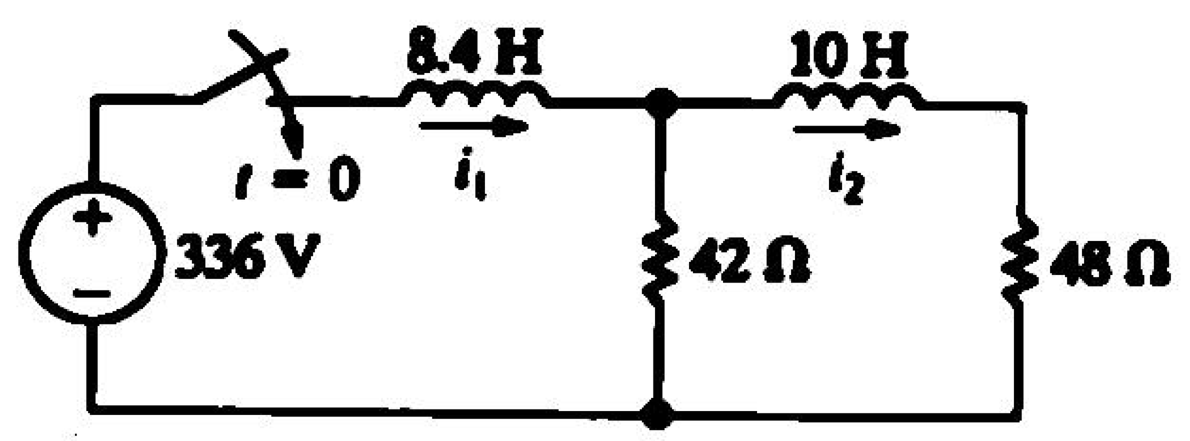
$$= e^{-0.6t}\left[160\cos 0.8t - 120\sin 0.8t\right]$$

$$A\cos x + B\sin x = \sqrt{A^2+B^2}\,\cos\left(x - \tan^{-1}\frac{B}{A}\right)$$

$$v(t) = e^{-0.6t}\sqrt{160^2+120^2}\,\cos\left(0.8t - \tan^{-1}\frac{-120}{160}\right)$$

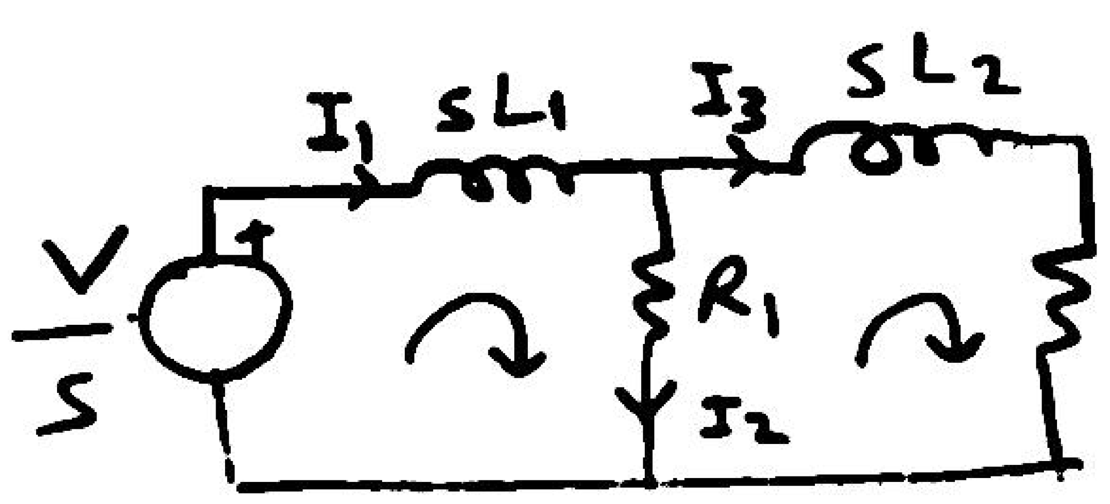
$$= 200\,e^{-0.6t}\cos(0.8t + 36.8)$$

The Step Response of a Multiple Mesh Circuit



Calculate currents and Voltages. initial currents are zero

Solution



$$-\frac{V}{s} + sL_1 I_1 + R_1 I_2 = 0 \qquad (1)$$

$$-R_1 I_2 + (sL_2 + R_2) I_3 = 0 \qquad (2)$$

$$I_1 = I_2 + I_3 \qquad (3)$$

3 equations 3 unknown.

$$(2) \rightarrow \quad I_2 = \frac{sL_2 + R_2}{R_1} I_3 = M\,I_3 \qquad (4)$$

$$(1)\text{ and }(3) \rightarrow \quad -\frac{V}{s} + sL_1(I_2 + I_3) + R_1 I_2 = 0 \qquad (5)$$

$$-\frac{V}{s} + sL_1(M I_3 + I_3) + R_1 M I_3 = 0$$

$$(sL_1 M + sL_1 + R_1 M) I_3 = \frac{V}{s}$$

$$I_3 = \frac{V}{s}\,\frac{1}{sL_1M + sL_1 + R_1M} = \frac{V}{s}\,\frac{1}{sL_1\,\frac{sL_2+R_2}{R_1} + sL_1 + R_1\,\frac{sL_2+R_2}{R_1}}$$

$$= \frac{V}{s}\,\frac{1}{\frac{L_1L_2}{R_1}s^2 + (L_1\frac{R_2}{R_1} + L_1 + L_2) + R_2}$$

$V = 336 \quad L_1 = 8.4 \quad L_2 = 10 \quad R_1 = 42 \quad R_2 = 48$

$$I_3 = \frac{336}{s}\,\frac{1}{2s^2 + 28s + 48} = \frac{336}{s}\,\frac{1/2}{s^2+14s+24}$$

$$I_3(s) = \frac{168}{s(s^2+14s+24)}$$

$$I_2(s) = M\,I_3 = \frac{sL_2+R_2}{R_1}\,\frac{168}{s(s^2+14s+24)} = \frac{40s+192}{s(s^2+14s+24)}$$

$I_1 = I_2 + I_3 = \qquad \Rightarrow \; i_1(t) = 15 - 14e^{-2t} - e^{-12t}$

$V_{L1} = sL_1\,I_1 =$

$V_{R1} = R_1\,I_2 =$

$V_{L2} = sL_2\,I_3 =$

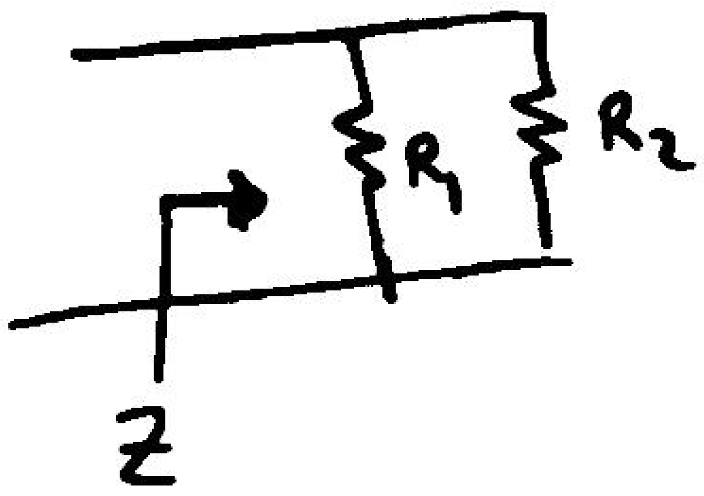
$V_{R2} = R_2\,I_3 =$

$i_3(t) = 7 - 8.4\,e^{-2t} + 1.4\,e^{-12t}$

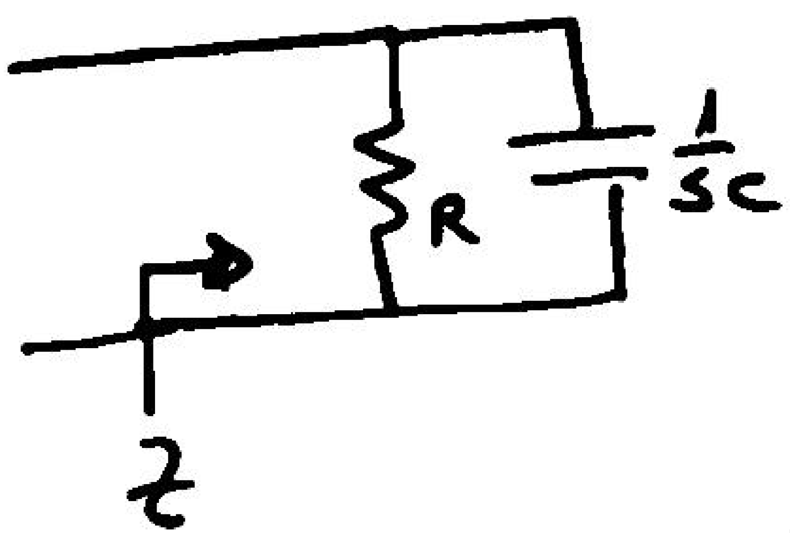
$i_2(t) = 8 - 5.6\,e^{-2t} - 2.4\,e^{-12t}$

Circuits in Laplace Transform.

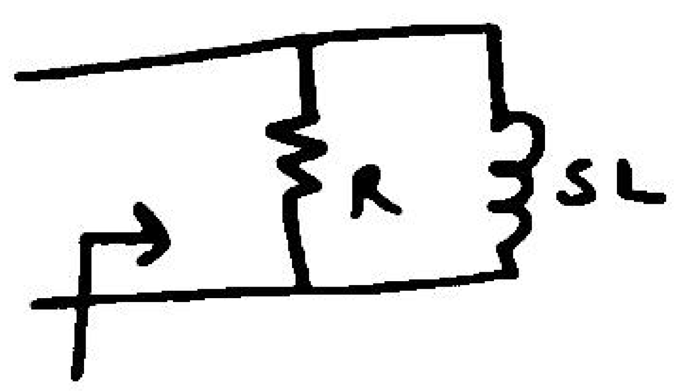
Calculate the impedances



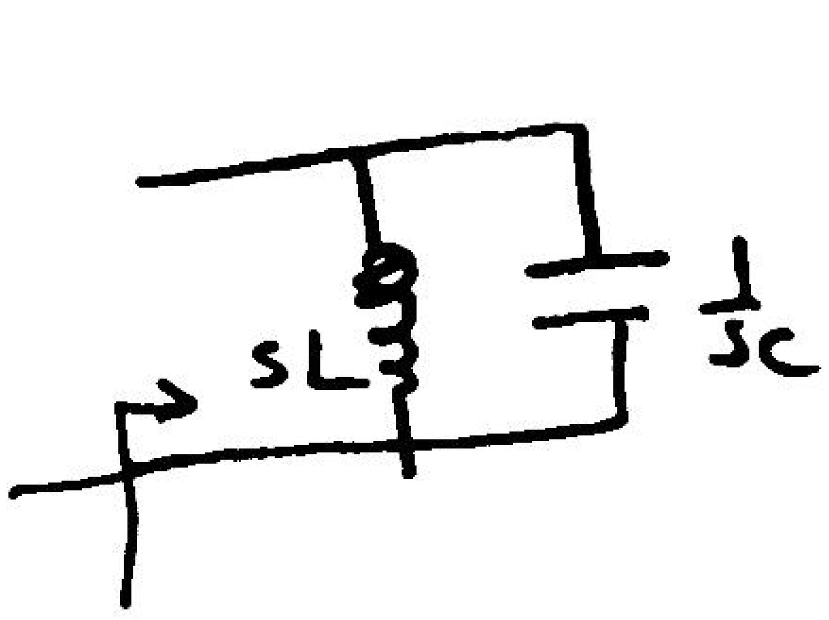
$$Z = \frac{R_1 R_2}{R_1 + R_2}$$



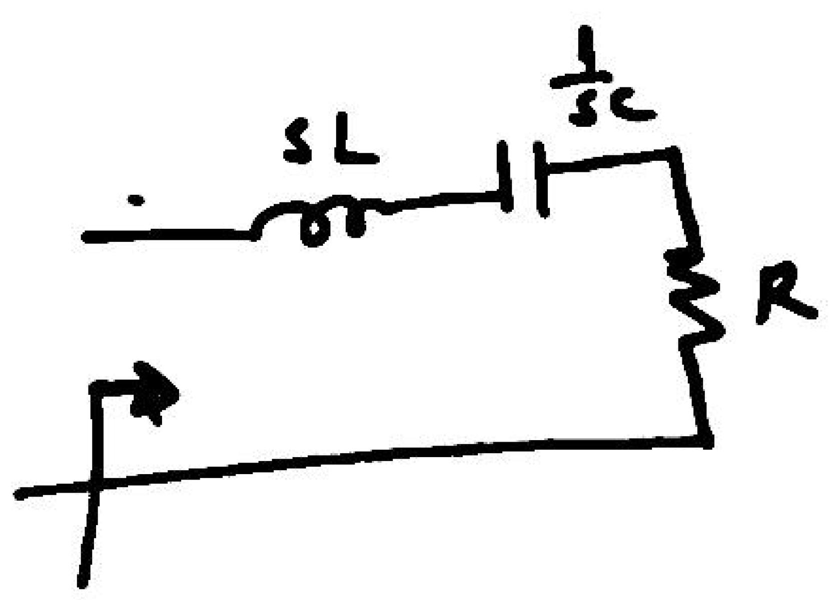
$$Z = \frac{R \cdot \frac{1}{sC}}{R + \frac{1}{sC}} = \frac{R}{sCR + 1}$$



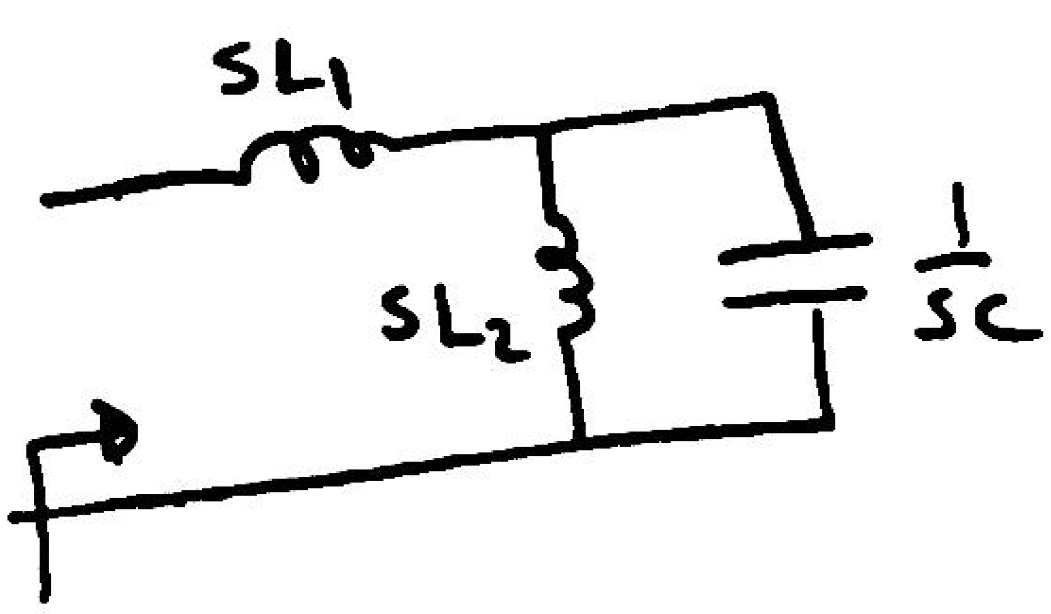
$$Z = \frac{R \cdot sL}{R + sL}$$



$$Z = \frac{sL \, \frac{1}{sC}}{sL + \frac{1}{sC}} = \frac{sL}{s^2LC + 1}$$



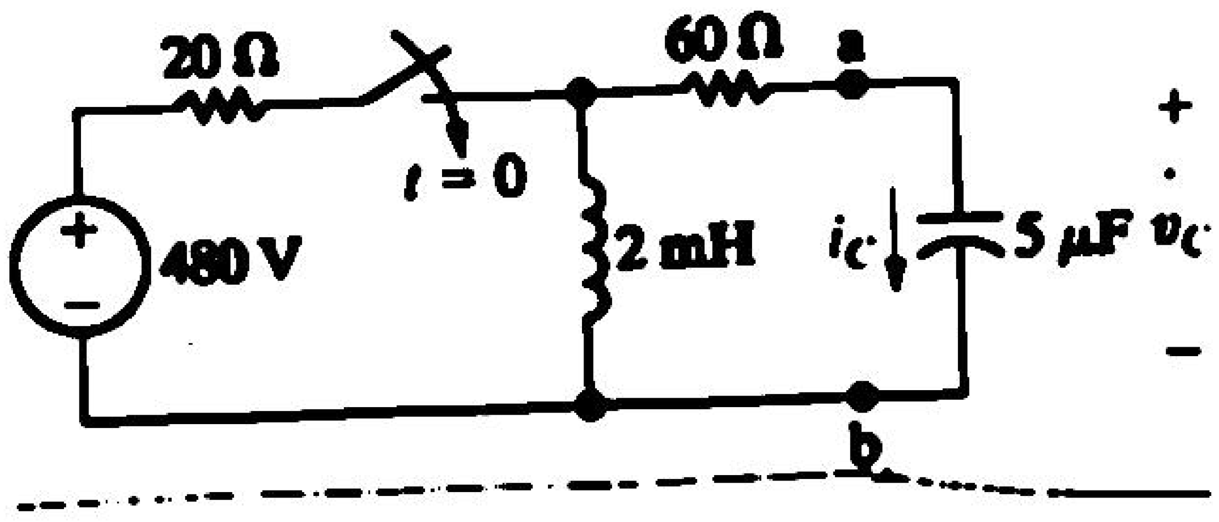
$$Z = sL + \frac{1}{sC} + R = \frac{s^2LC + 1 + sCR}{sC}$$



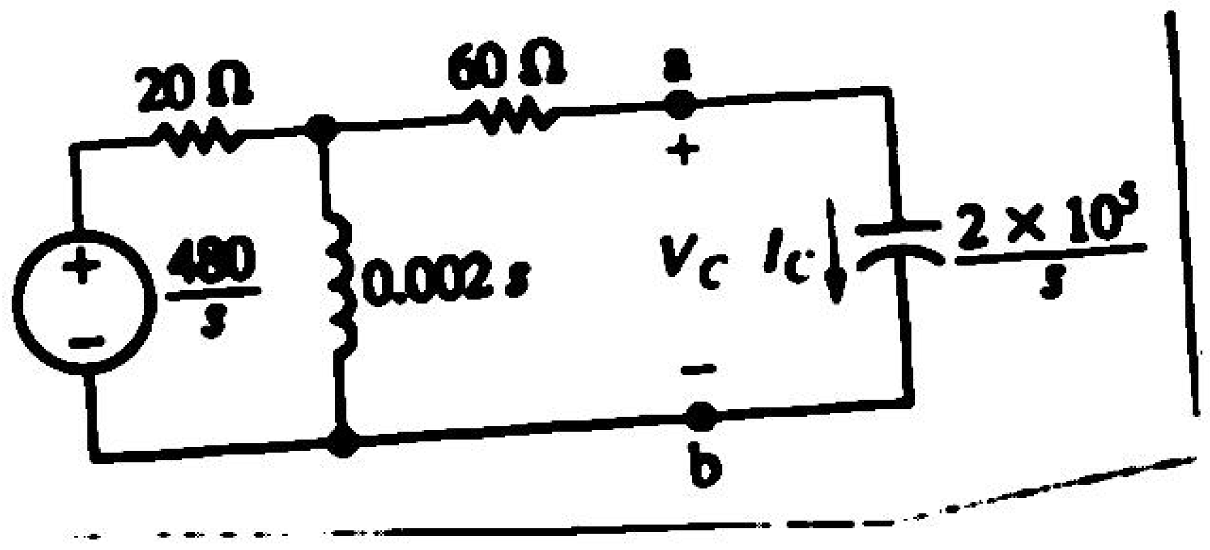
$$Z = sL_1 + \frac{sL_2 \, \frac{1}{sC}}{sL_2 + \frac{1}{sC}} = sL_1 + \frac{sL_2}{s^2L_2C + 1}$$

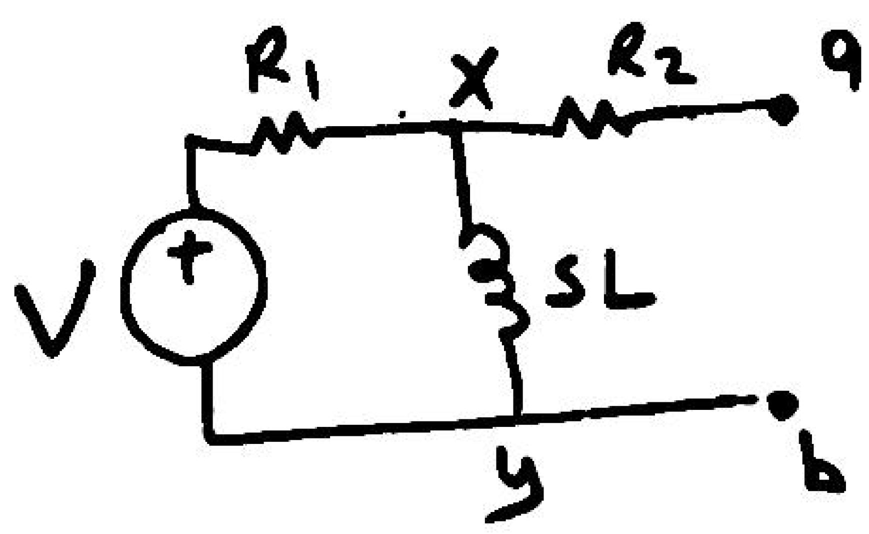
$$= \frac{s^3L_1L_2C + sL_1 + sL_2}{s^2L_2C + 1}$$

Calculate Thevenin equivalent of the following circuit. $I_L(0)=0$ $V_c(0)=0$

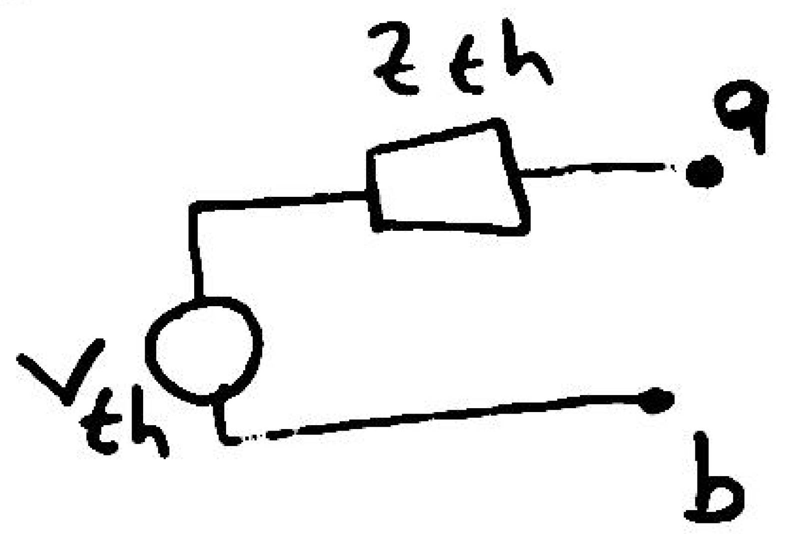


Solution

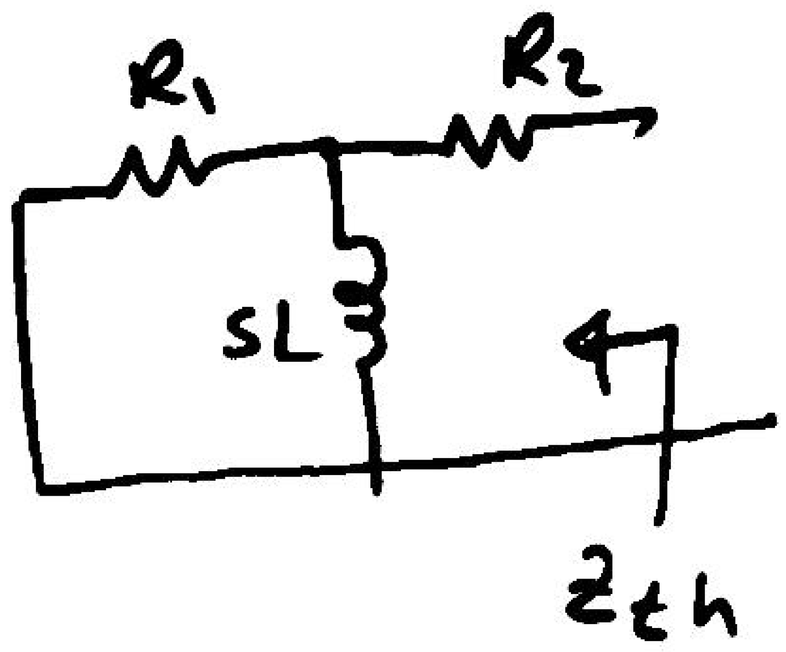




$\equiv$



$$V_{th} = V_{ab} = V_{xy} = V \frac{SL}{SL+R_1} = \frac{480}{s} \frac{s \times 0.002}{s \times 0.002 + 20} = \frac{480}{s+10000}$$

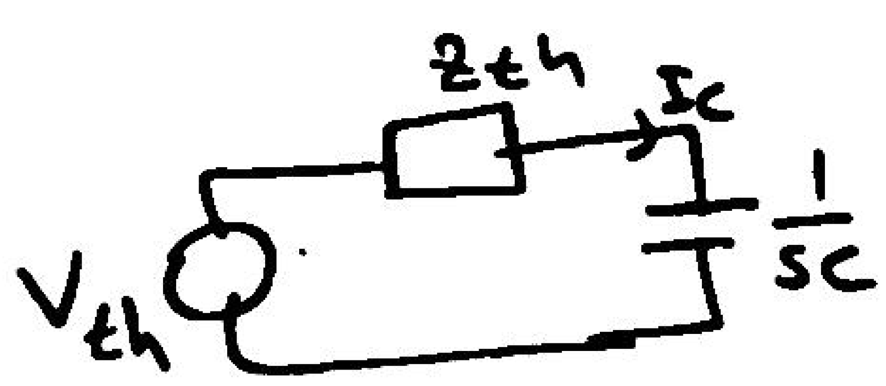


$$Z_{th} = R_2 + \frac{SL\ R_1}{SL+R_1} = \frac{R_2 SL + R_2 R_1 + SL\ R_1}{SL + R_1}$$

$$= \frac{60 \times s \times 0.002 + 60 \times 20 + s\ 0.002 \times 20}{s \times 0.002 + 20}$$

$$= \frac{80(s+7500)}{s+10\,000}$$

Calculate I_c if capacitor is connected



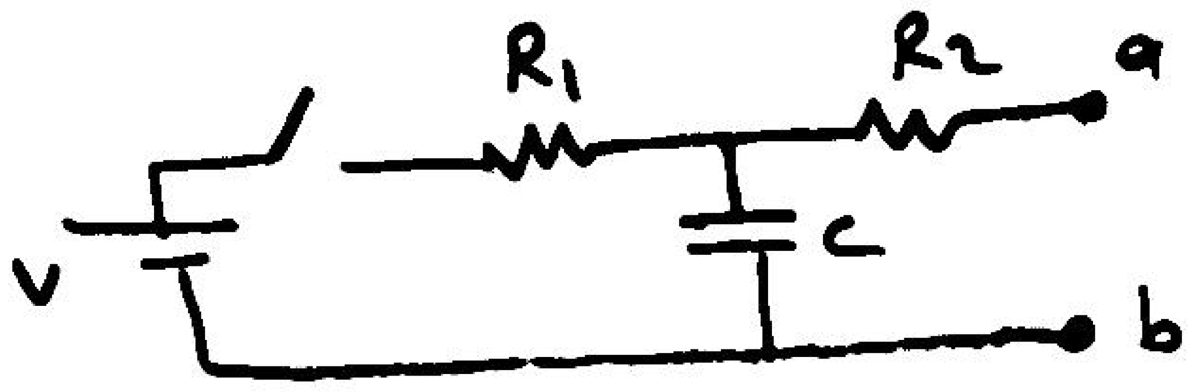
$$I_c = \frac{V_{th}}{Z_{th} + \frac{1}{sc}} = \frac{\frac{4800}{s+10000}}{\frac{80(s+7500)}{s+10000} + \frac{1}{s \times 5 \times 10^{-6}}}$$

$$= \frac{4800}{80(s+7500) + \frac{s+1000}{s \times 5.10^{-6}}} = \frac{6s}{(s+5000)^2}$$

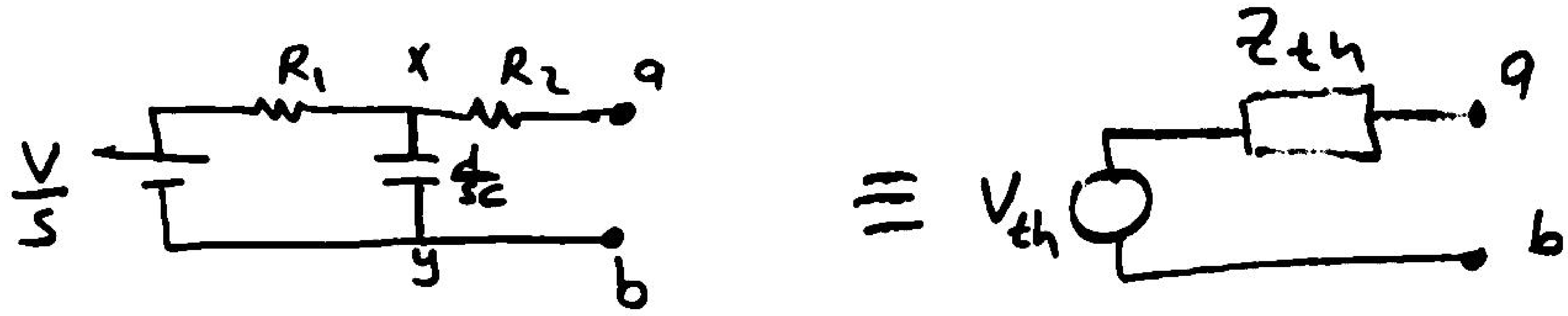
$$I_c(s) = \frac{6}{s+5000} - \frac{30000}{(s+5000)^2} \rightarrow i_c(t) = 6e^{-5000t} - 30000\, t\, e^{-5000t}$$

calculate thevenin equivalent circuit.

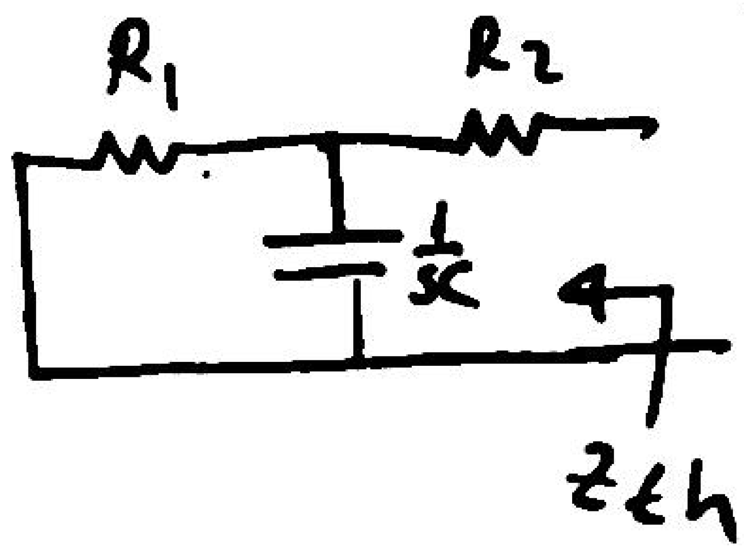
$V_c(0)=0$



solution



$$V_{ab}=V_{th}=V_{xy}=\frac{V}{s}\,\frac{\frac{1}{sC}}{\frac{1}{sC}+R_1}=\frac{V}{s}\,\frac{1}{sCR+1}$$

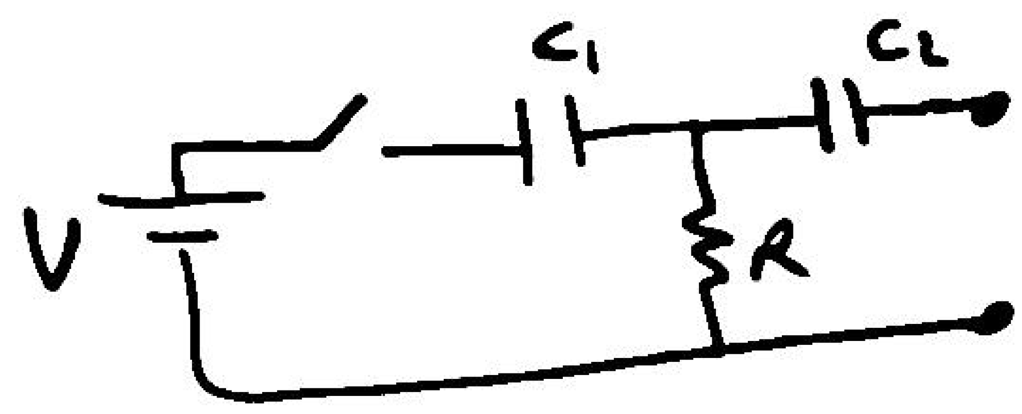


$$Z_{th}=R_2+\frac{R_1\,\frac{1}{sC}}{R_1+\frac{1}{sC}}=R_2+\frac{R_1}{R_1sC+1}$$

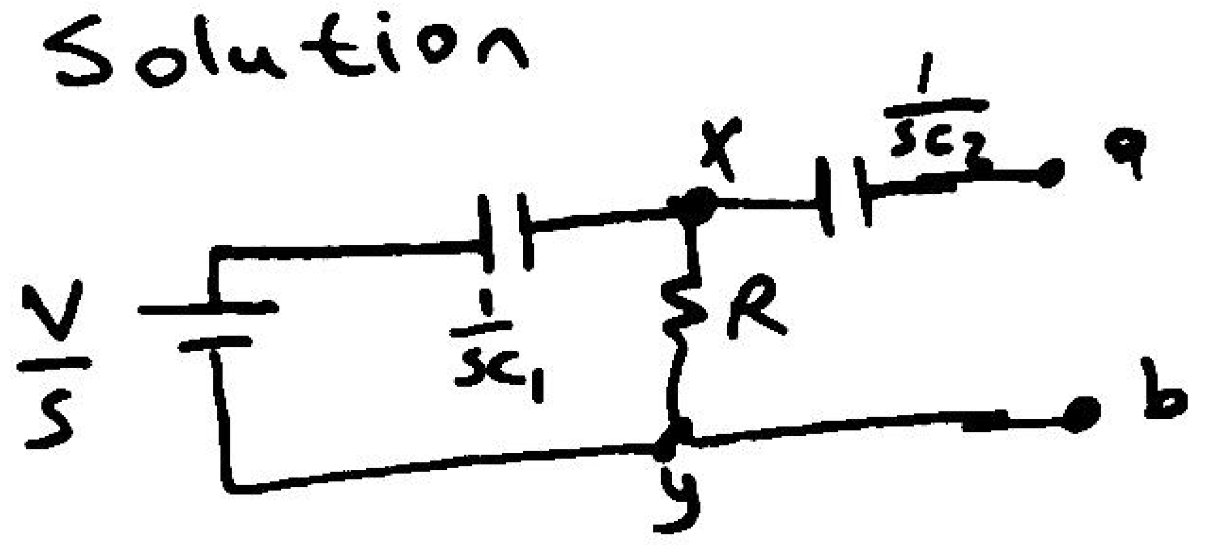
$$Z_{th}=\frac{R_2R_1sC+R_2+R_1}{R_1sC+1}$$

---

calculate thevenin equivalent circuit



Solution

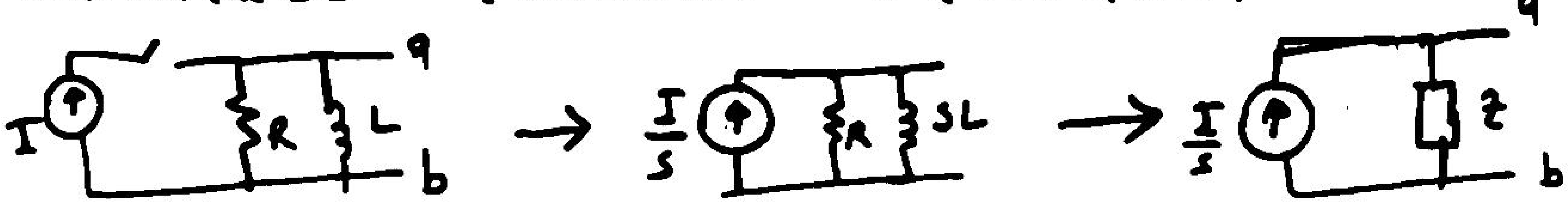


$$V_{ab}=V_{xy}=\frac{V}{s}\,\frac{R}{R+\frac{1}{sC_1}}=\frac{V}{s}\,\frac{sC_1R}{sC_1R+1}=\frac{RC_1V}{sC_1R+1}$$

$$Z_{th}=\frac{1}{sC_2}+\frac{R\cdot\frac{1}{sC_1}}{R+\frac{1}{sC_1}}=\frac{1}{sC_2}+\frac{R}{RsC_1+1}$$

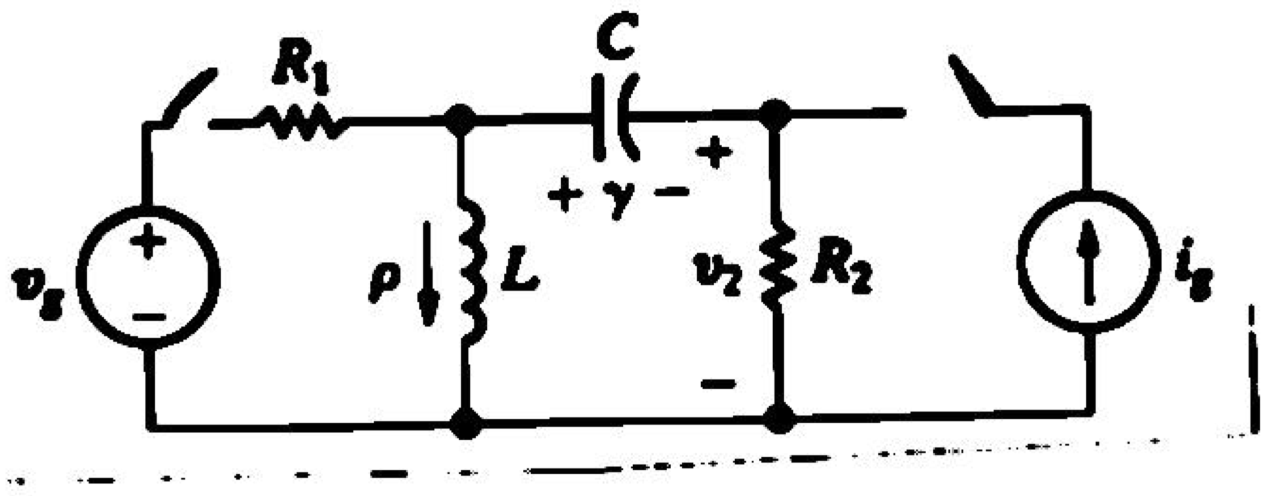
---

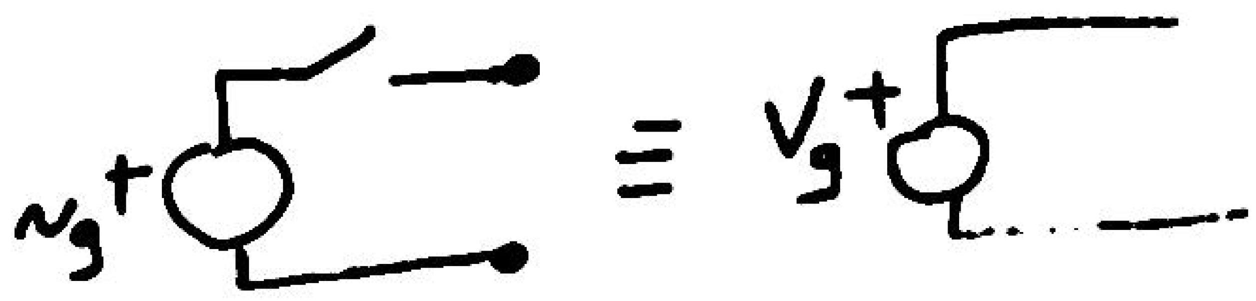
calculate thevenin equivalent



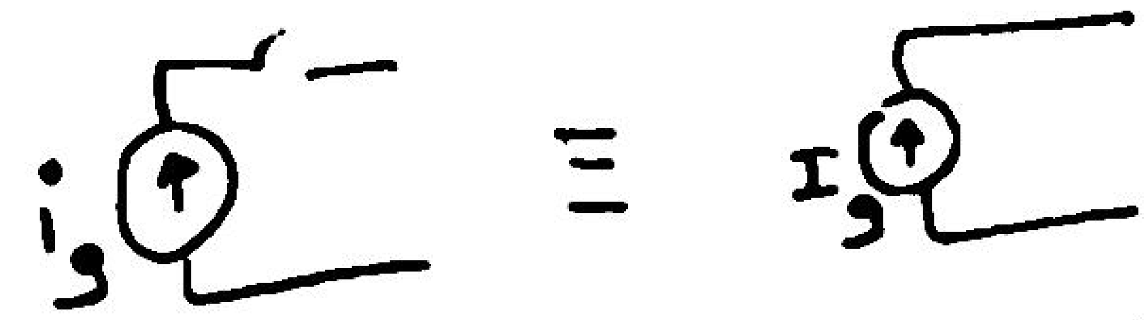
$$Z=\frac{sLR}{sL+R}$$

$$V_{th}=\frac{I}{s}\times Z=\frac{I}{s}\,\frac{sLR}{sL+R}\qquad Z_{th}=Z=\frac{sLR}{sL+R}$$

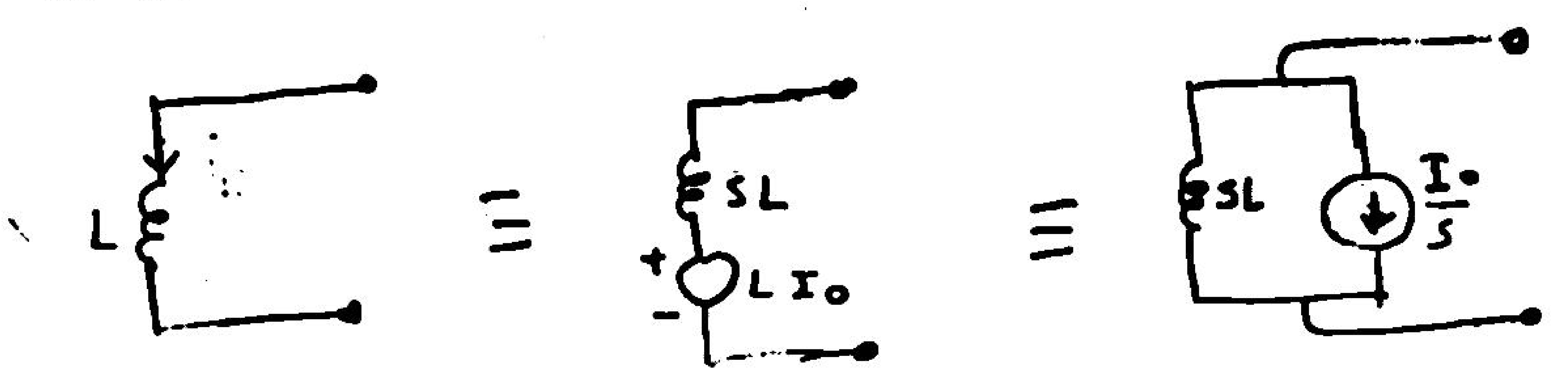


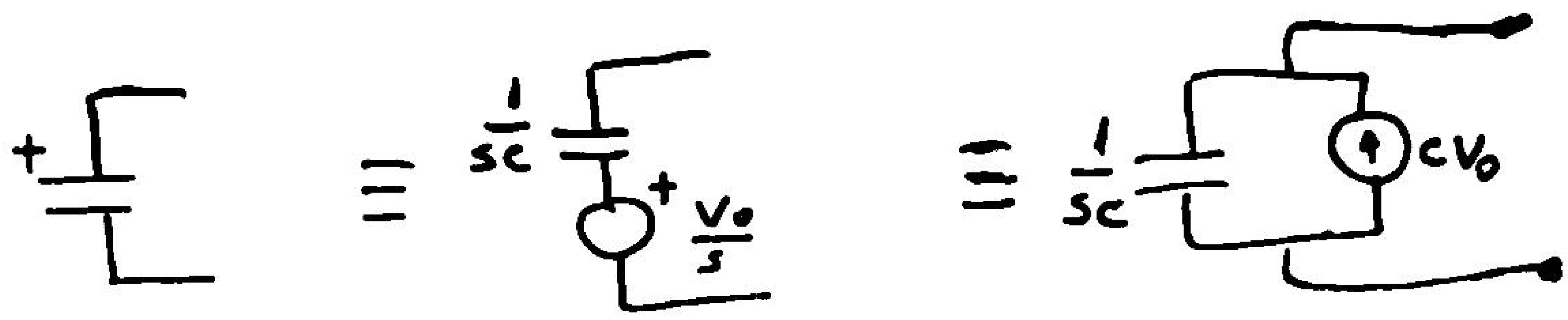


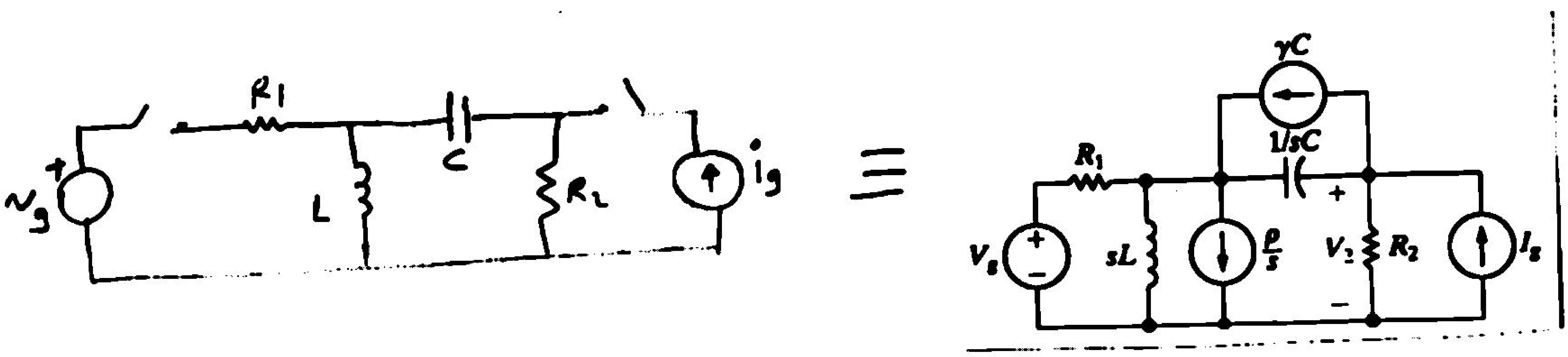
if $v_g = A\,u(t) \longrightarrow V_g = \frac{A}{s}$



if $i_g = B\,u(t) \longrightarrow I_g = \frac{B}{s}$

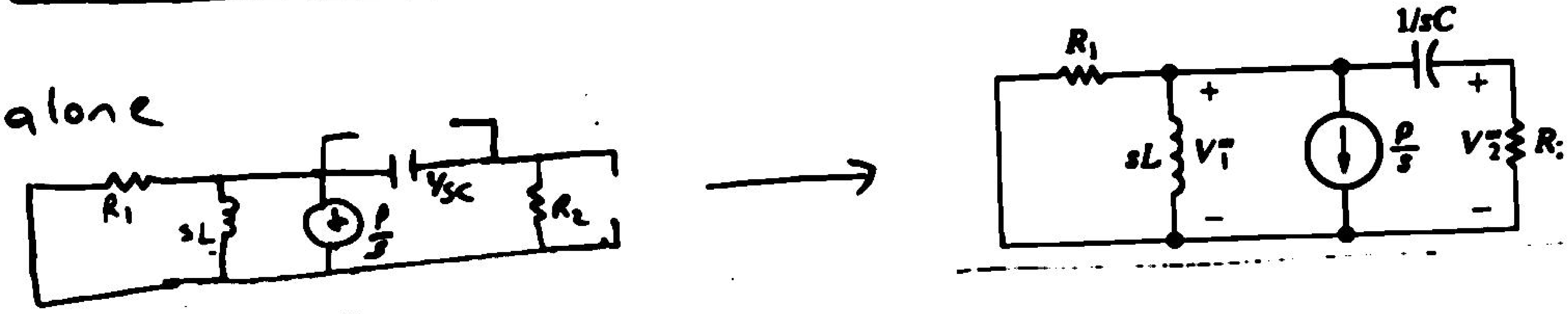




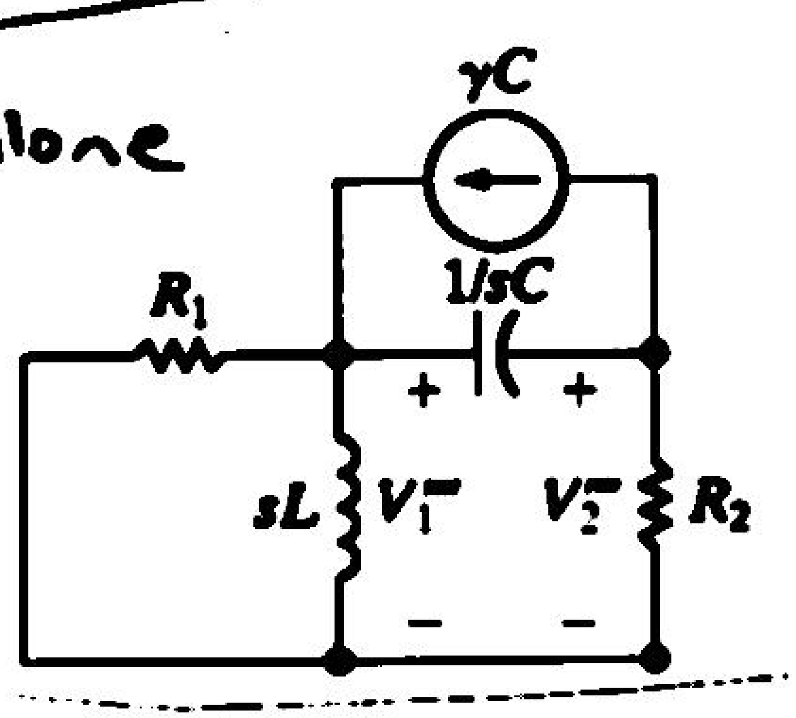


1) V_g alone

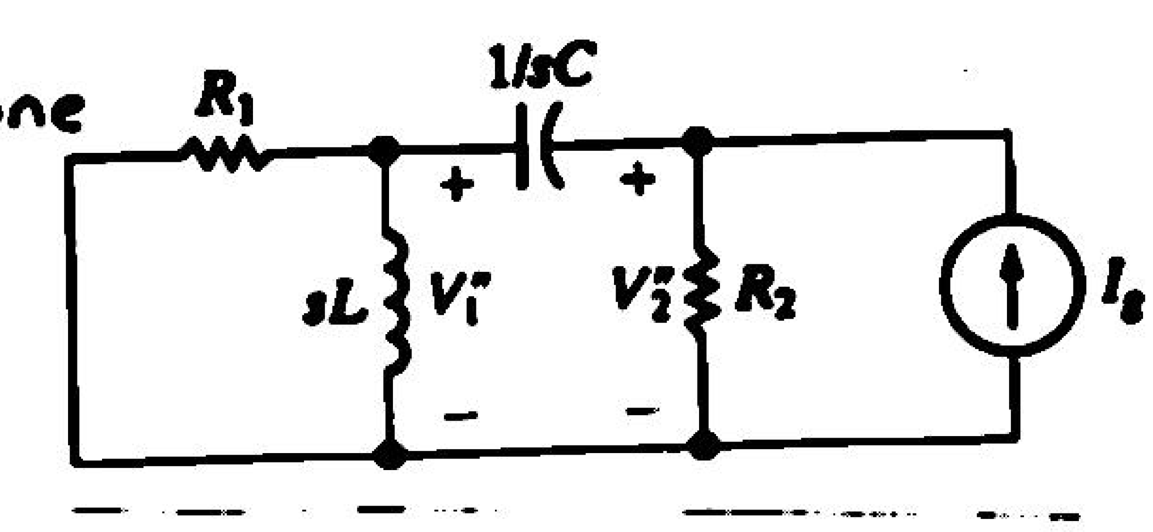
2) $\frac{\rho}{s}$ alone



3) γC alone



4) I_g alone

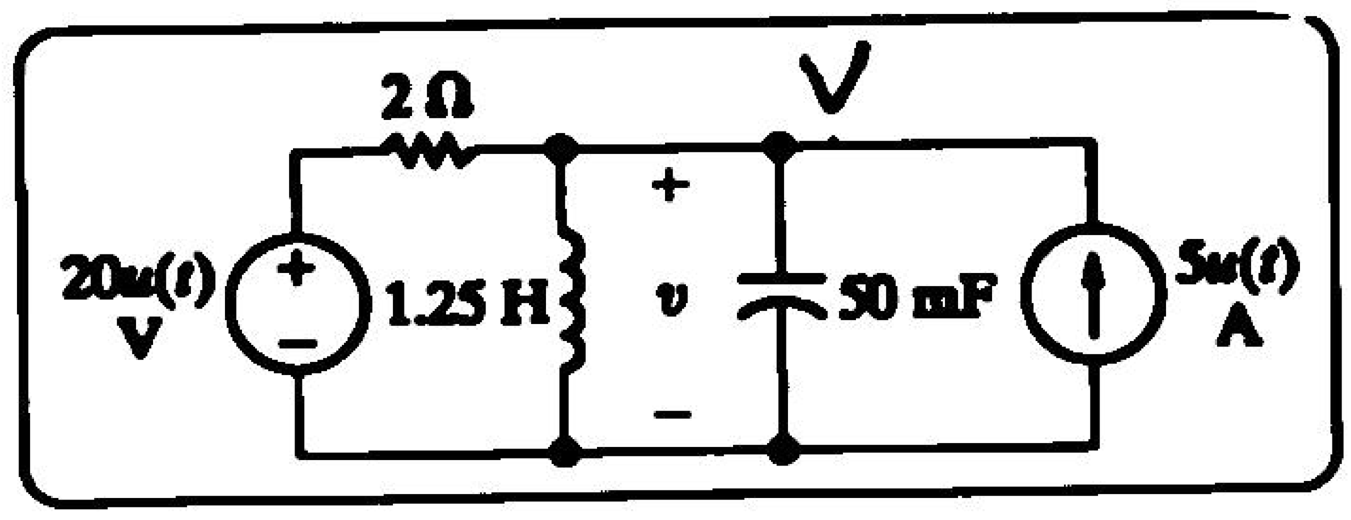


13.8 The energy stored in the circuit shown is zero at the instant the two sources are turned on.

a) Find the component of v for $t > 0$ owing to the voltage source.

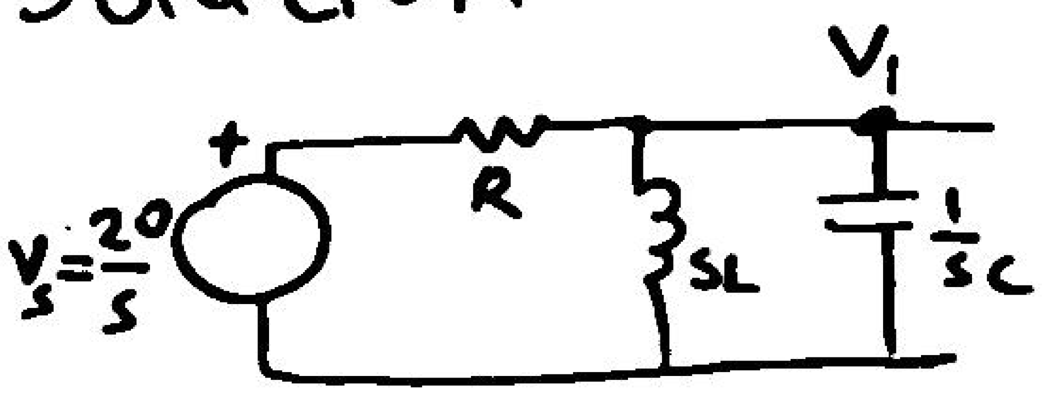
b) Find the component of v for $t > 0$ owing to the current source.

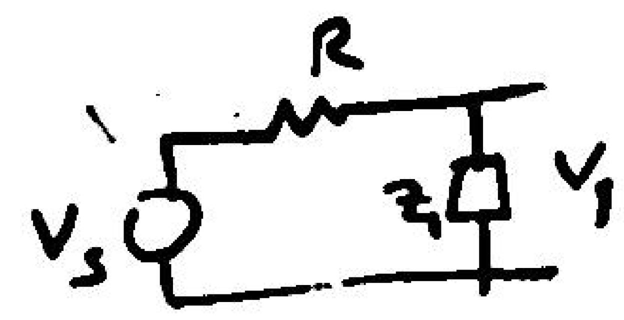
c) Find the expression for v when $t > 0$.



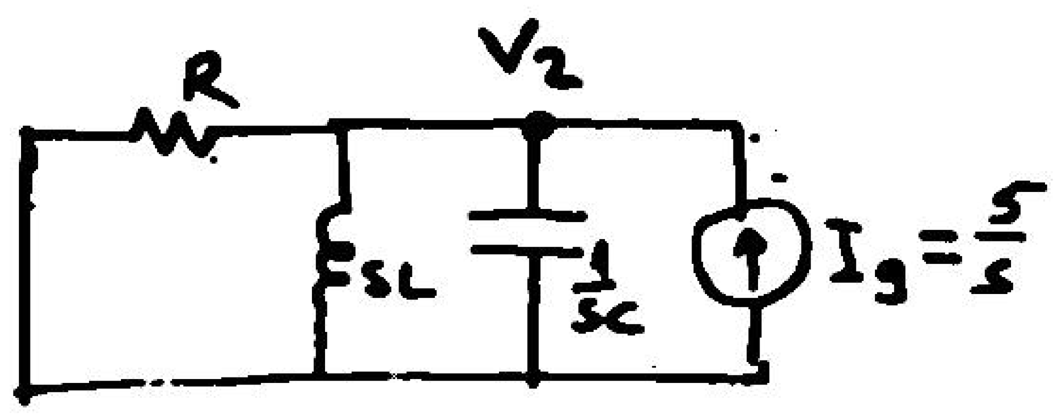
ANSWER: (a) $[(100/3)e^{-2t} - (100/3)e^{-8t}]u(t)$ V;
(b) $[(50/3)e^{-2t} - (50/3)e^{-8t}]u(t)$ V;
(c) $[50e^{-2t} - 50e^{-8t}]u(t)$ V.

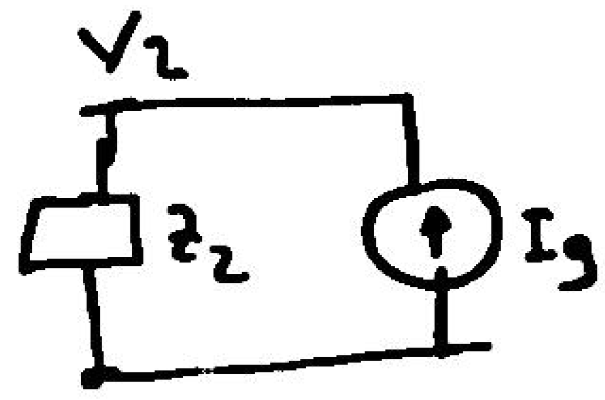
Solution





$$V_1 = V_s \frac{Z_1}{Z_1 + R} \qquad Z_1 = \frac{SL \frac{1}{SC}}{SL + \frac{1}{SC}} = \frac{SL}{S^2LC+1}$$





$$V_2 = Z_2 I_g \qquad \frac{1}{Z_2} = \frac{1}{R} + \frac{1}{SL} + \frac{1}{\frac{1}{SC}}$$

$$V = V_1 + V_2$$

calculations

$$Z_1 = \frac{SL}{S^2LC+1} = \frac{1.25\,S}{S^2\ 1.25\ 50\ 10^{-3} + 1} = \frac{1.25\,S}{0.0625\,S^2+1} = \frac{20\,S}{S^2+16}$$

$$V_1 = V_S \frac{Z_1}{Z_1 + R} = \frac{20}{S} \frac{\frac{20\,S}{S^2+16}}{\frac{20\,S}{S^2+16} + 2} = \frac{20}{S} \frac{20\,S}{20S + 2(S^2+16)} = \frac{400}{2S^2+20S+32}$$

$$\frac{1}{Z_2} = \frac{1}{2} + \frac{1}{1.25\,S} + S\ 50\ 10^{-3} = 0.5 + \frac{0.8}{S} + 0.05\,S = \frac{0.05\,S^2+0.5\,S+0.8}{S}$$

$$Z_2 = \frac{S}{0.05\,S^2+0.5\,S+0.8} = \frac{20\,S}{S^2+10S+16}$$

$$V_2 = Z_2 I_g = \frac{20\,S}{S^2+10S+16} \cdot \frac{5}{S} = \frac{100}{S^2+10\,S+16}$$

$$V = V_1 + V_2 = \frac{400}{2(S^2+10S+16)} + \frac{100}{S^2+10\,S+16} = \frac{300}{S^2+10S+16} = \frac{A}{(S+8)} + \frac{B}{S+2}$$

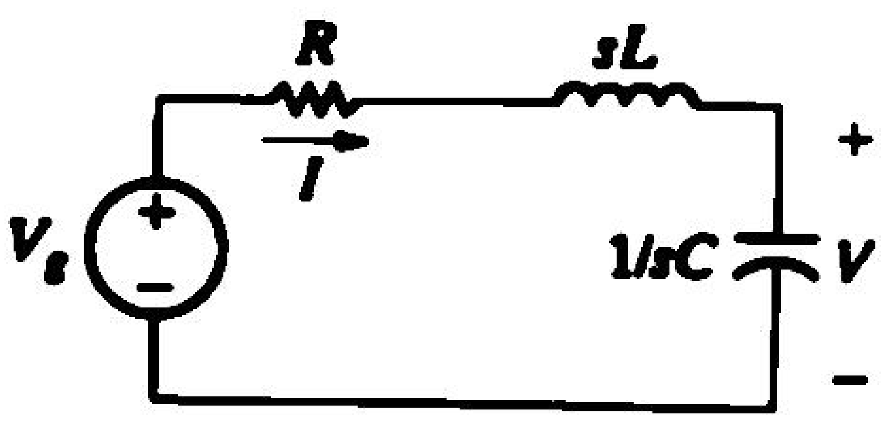
$A = -50 \qquad B = 50$

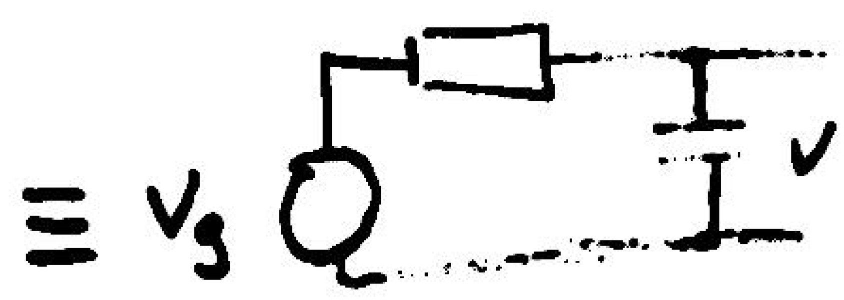
$$v(t) = -50\,e^{-8t} + 50\,e^{-2t}$$

13.4 ◆ The Transfer Function

$$H(s) = \frac{Y(s)}{X(s)}, \tag{13.93}$$

. $Y(s)$ is the Laplace transform of the output signal, and $X(s)$ is the Laplace transform of the input signal.





$$H(s) = \frac{V}{V_g} = \frac{1/sC}{R + sL + 1/sC} = \frac{1}{s^2LC + RCs + 1}$$

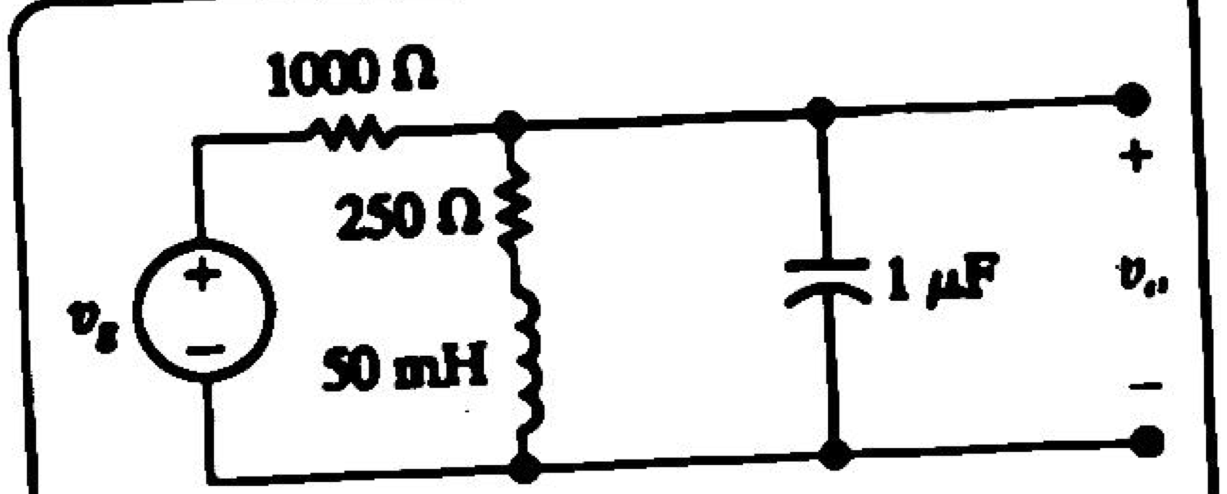
[redacted] A series RLC circuit.

EXAMPLE 13.1

The voltage source v_g drives the circuit shown in Fig. 13.31. The response signal is the voltage across the capacitor, v_o.

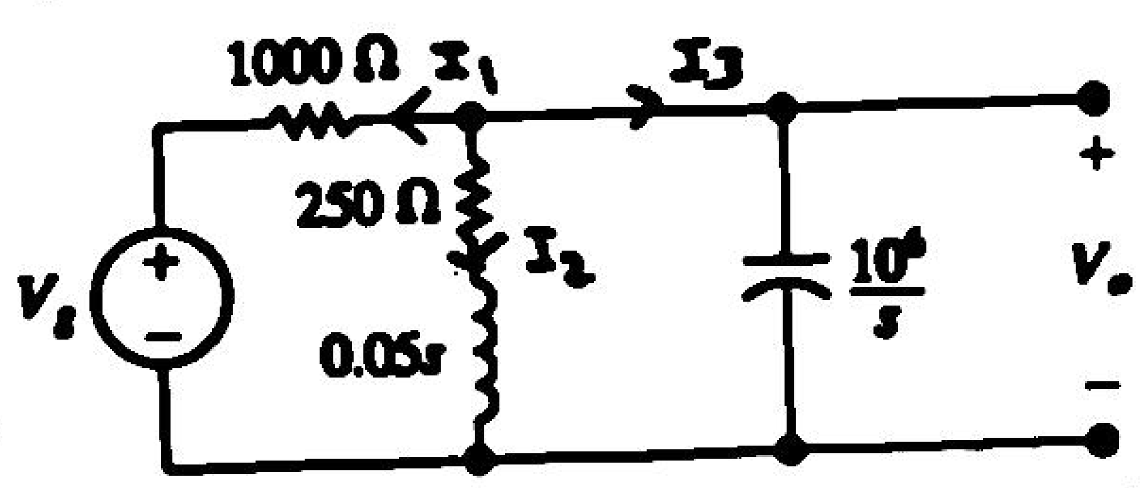
a) Calculate the numerical expression for the transfer function.

b) Calculate the numerical values for the poles and zeros of the transfer function.



[redacted] The circuit for Example 13.1.

SOLUTION



$$I_1 + I_2 + I_3 = 0$$

$$\frac{V_o - V_g}{1000} + \frac{V_o}{250 + 0.05s} + \frac{V_o s}{10^6} = 0.$$

$$V_o\left(\frac{1}{1000} + \frac{1}{250 + 0{,}05s} + \frac{s}{10^6}\right) - \frac{V_g}{1000} = 0$$

$$V_o = \frac{V_g}{1000} \cdot \frac{1}{\frac{1}{1000} + \frac{1}{250 + 0{,}05s} + \frac{s}{10^6}}$$

$$V_o = \frac{1000(s + 5000)V_g}{s^2 + 6000s + 25 \times 10^6}.$$

Hence the transfer function is

$$H(s) = \frac{V_o}{V_g}$$

$$= \frac{1000(s + 5000)}{s^2 + 6000s + 25 \times 10^6}.$$

b) The poles of $H(s)$ are the roots of the denominator polynomial. Therefore

$$-p_1 = -3000 - j4000,$$

$$-p_2 = -3000 + j4000.$$

The zeros of $H(s)$ are the roots of the numerator polynomial; thus $H(s)$ has a zero at

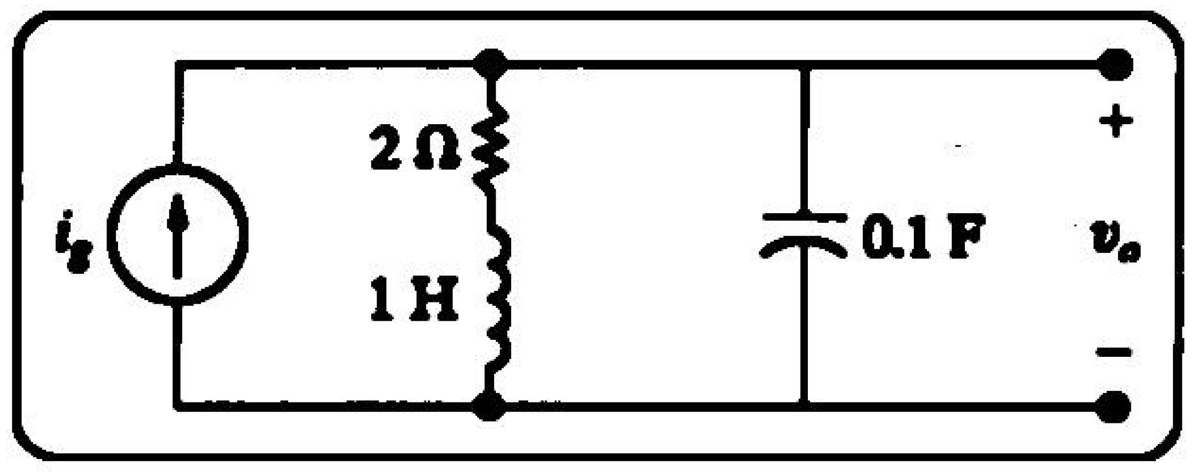
$$-z_1 = -5000.$$

Poles = Denominator roots

Zeros = Numerator roots

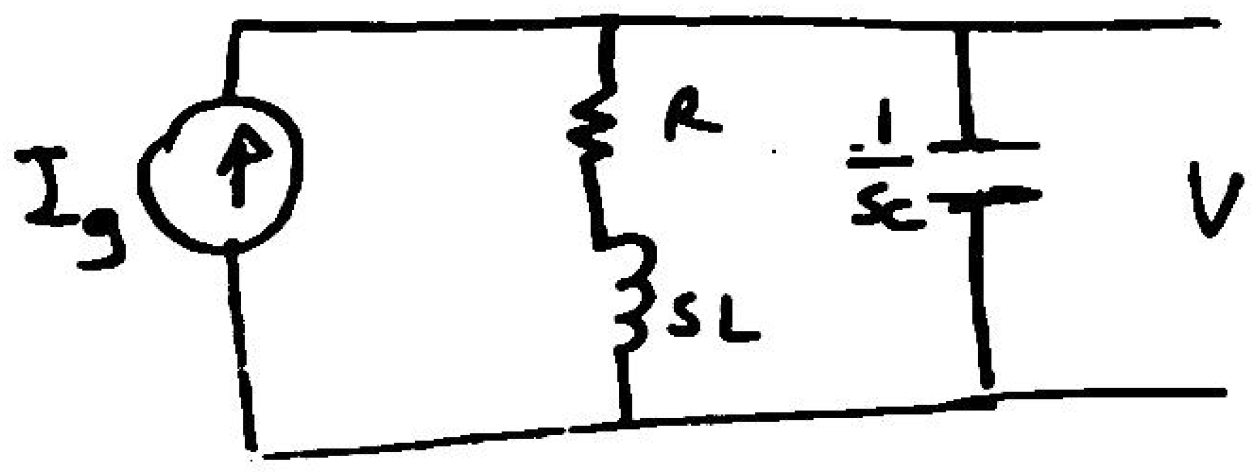
13.8 a) Derive the numerical expression for the transfer function V_o/I_g for the circuit shown.

b) Give the numerical value of each pole and zero of $H(s)$.



ANSWER: (a) $H(s) = 10(s+2)/(s^2+2s+10)$;
(b) $-p_1 = -1 + j3$, $-p_2 = -1 - j3$, $-z = -2$.

solution



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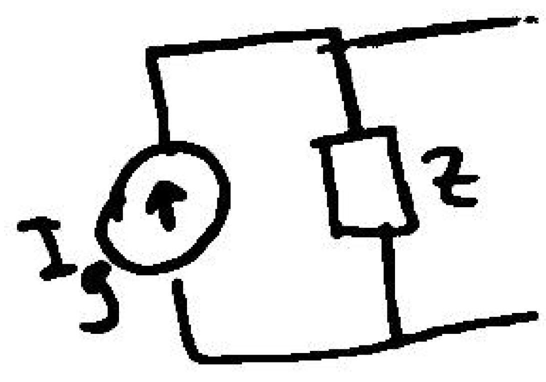


$V_o = Z I_g$

$$V_o = Z I_g \rightarrow \frac{V_o}{I_g} = Z \qquad H(s) = \frac{V_o}{I_g} = Z$$

$$Z = (R+SL) // \frac{1}{SC} = \frac{(R+SL)\frac{1}{SC}}{(R+SL) + \frac{1}{SC}} = \frac{R+SL}{(R+SL)SC+1}$$

$$= \frac{R+SL}{s^2 LC + SCR + 1} = \frac{2 + S \times 1}{s^2\, 1 \times 0.1 + S \times 0.1 \times 2 + 1} = \frac{s+2}{0.1 s^2 + 0.2 s + 1}$$

$$= \frac{10(s+2)}{s^2 + 2s + 10}$$

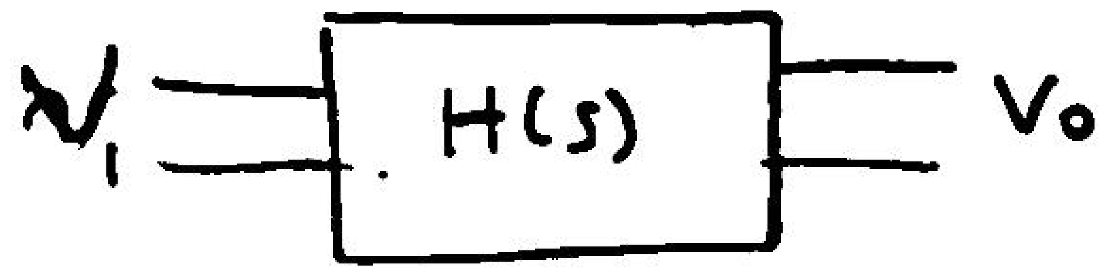
$$H(s) = Z = \frac{10(s+2)}{s^2 + 2s + 10}$$

Poles $\rightarrow$ $s^2 + 2s + 10 = 0$ $\qquad S_{12} = \dfrac{-2 \mp \sqrt{2^2 - 40}}{2} = -1 \mp 3j$

$s_1 = -1 + 3j \qquad s_2 = -1 - 3j$

Zeros $\rightarrow$ $10(s+2) = 0$ $\Rightarrow s = -2$

Example problem: $H(s) = \dfrac{s+4}{s^2+3s+2}$



calculate $v_o(t)$ for

a) $v_1(t) = u(t)$ b) $v_1(t) = \cos 5t$

Solution

$$H(s) = \frac{V_o}{V_1} \Rightarrow V_o = H(s) \cdot V_1$$

a) $v_1(t) = u(t) \longrightarrow V_1(s) = \frac{1}{s}$ $\quad V_o = H(s)\, V_1(s) = \frac{1}{s} \, \frac{s+4}{s^2+3s+2}$

$$s^2+3s+2 = 0 \begin{array}{l} \nearrow s_1 = -2 \\ \searrow s_2 = -1 \end{array}$$

$$V_o = \frac{s+4}{s\,(s^2+3s+2)} = \frac{s+4}{s\,(s+2)(s+1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1}$$

$$A = 2 \qquad B = 1 \qquad C = -3$$

$$V_o(s) = \frac{2}{s} + \frac{1}{s+2} - \frac{3}{s+1} \longrightarrow v_o(t) = 2\,u(t) + e^{-2t} - 3e^{-t}$$

b) $v_1(t) = \cos 5t \longrightarrow V_1(s) = \dfrac{s}{s^2+5^2}$

$$V_o = H(s)\,V_1(s) = \frac{s+4}{s^2+3s+2} \cdot \frac{s}{s^2+5^2} = \frac{s^2+4s}{(s+2)(s+1)(s+5i)(s-5i)}$$

$$= \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{s+5i} + \frac{D}{s-5i}$$

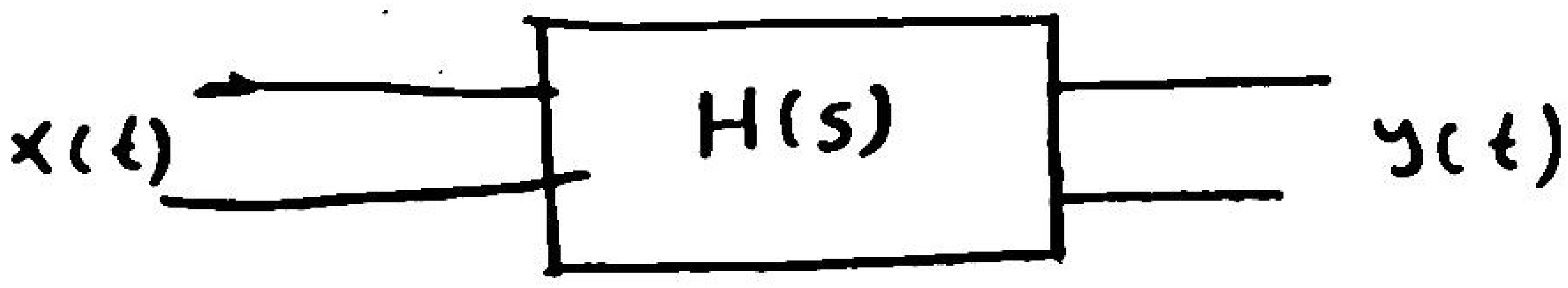
$$A = 0.1034 \qquad B = -0.077 \qquad \begin{array}{l} C = -0.0133 + 0.113j \\ D = -0.0133 - 0.113j \end{array}$$

$$V_o(s) = \frac{0.1034}{s+2} - \frac{0.077}{s+1} + \frac{-0.0133+0.113j}{s+5j} + \frac{-0.0133-0.113j}{s-5j}$$

$$\mathcal{L}\left\{\frac{a+bi}{s-(x+iy)} + \frac{a-bi}{s-(x-iy)}\right\} = e^{xt}\left\{2a\cos yt - 2b\sin yt\right\}$$

$$\mathcal{L}\left\{\frac{-0.0133+0.113j}{s-(0-5j)} + \frac{-0.0133-0.113j}{s-(0+5j)}\right\} = e^{0}(2(-0.013\cos 5t - 2(0.11)\sin 5t$$

$$v_o(t) = 0.1034\,e^{-2t} - 0.077\,e^{-t} - 2 \times 0.013\cos 5t - 2 \times 0.11\sin 5t$$



$x(t) = A\cos(wt+\phi)$ $\qquad$ $y(t) = A\,|H(jw)|\cos(wt+\phi+\theta(w))$

$\theta(w) = \angle H(jw)$

Example problem: $H(s) = \frac{s+1}{s+2}$ $\quad x(t) = 8\cos 3t$ $\quad y(t) = ?$

Solution

$H(s) = \frac{s+1}{s+2} \longrightarrow H(jw) = \frac{jw+1}{jw+2}$ $\quad w=3 \rightarrow H(jw) = \frac{j3+1}{j3+2}$

$$|H(jw)|_{w=3} = |H(j3)| = \frac{\sqrt{3^2+1^2}}{\sqrt{3^2+2^2}} = 0{,}95$$

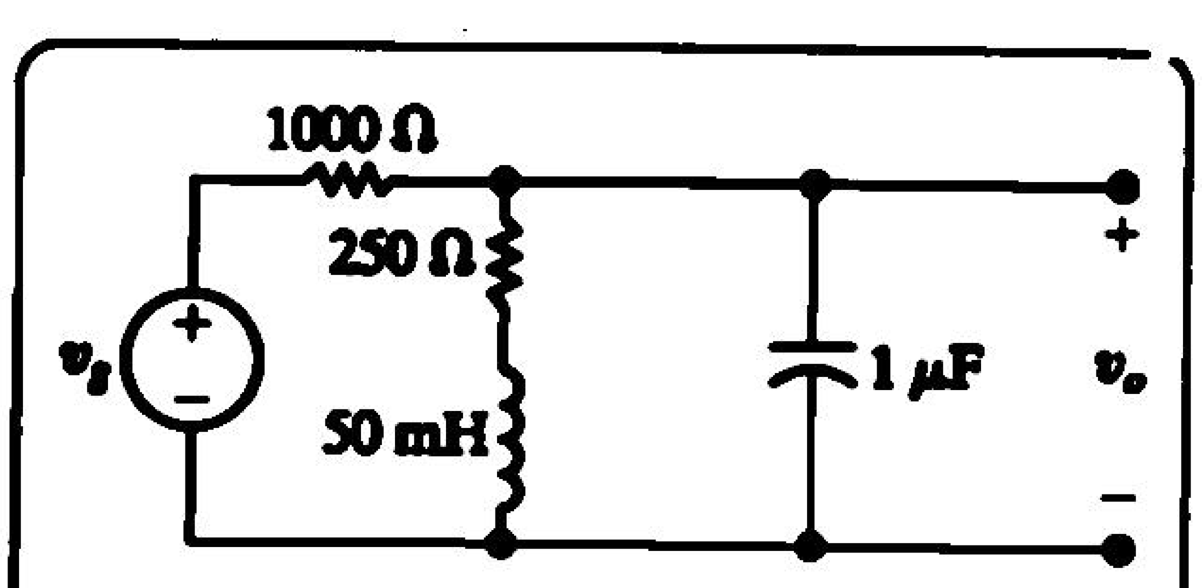
$$\angle H(jw)|_{w=3} = \angle H(j3) = \tan^{-1}\frac{3}{1} - \tan^{-1}\frac{3}{2} = 71.5 - 56.3 = 15{,}2$$

$$x(t) = 8\cdot\cos 3t \longrightarrow y(t) = 8\times 0.95\cos(3t+15{,}2) = 7{,}6\cos(3t+15{,}2)$$

$$H(jw) = H(s)|_{s=jw}$$

EXAMPLE 13.4

The circuit from Example 13.1 is shown in Fig. 13.46. The sinusoidal source voltage is $120\cos(5000t + 30^\circ)$ V. Find the steady-state expression for v_o.



The circuit for Example 13.4.

SOLUTION

From Example 13.1,

$$H(s) = \frac{1000(s+5000)}{s^2 + 6000s + 25 * 10^6}.$$

The frequency of the voltage source is 5000 rad/s; hence we evaluate $H(s)$ at $H(j5000)$:

$$H(j5000) = \frac{1000(5000 + j5000)}{-25 * 10^6 + j5000(6000) + 25 \times 10^6}$$

$$= \frac{1+j1}{j6} = \frac{1-j1}{6} = \frac{\sqrt{2}}{6}\angle{-45^\circ}.$$

Then, from Eq. 13.120,

$$v_{o_{ss}} = \frac{(120)\sqrt{2}}{6}\cos(5000t + 30^\circ - 45^\circ)$$

$$= 20\sqrt{2}\cos(5000t - 15^\circ)\text{ V}.$$