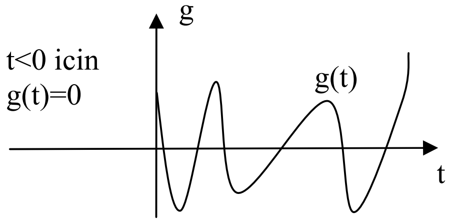
Laplas Donusumu

₤{g(t)}=G(s)

$₤^{-1}$ { G(s) }= g(t)

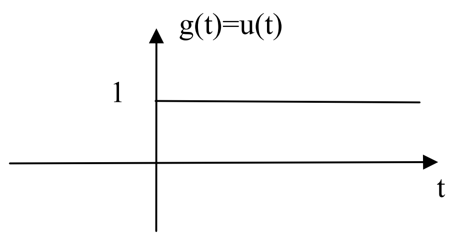
$$G(s) = \int_{t=0}^{\infty} g(t)e^{-st}dt$$

Laplas donusumu alinan fonksiyonlarin negatif kismi sifir varsayilir. yani t<0 icin g(t)=0 dir. Integralalinirken s degiskeni cok buyuk ve pozitif bir sayi varsayilir.



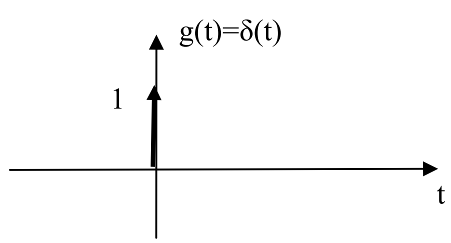
Muendislikte cok kullanilan fonksiyonlar

1)Birim basamak fonksiyonu u(t). t<0 icin g(t)=0, ve t>0 icin g(t)=1 olan fonksiyondur.



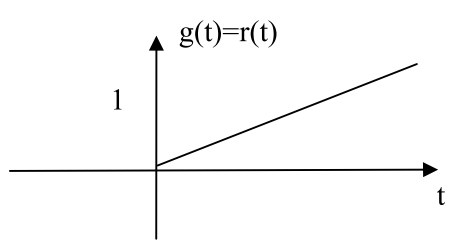
2)Impuls fonksiyonu δ(t). t≠0 icin g(t)=0, ve

$\int_{t=0}^{\infty} \delta(t)dt = 1$ bagintisi varsayilir.



Muendislikte cok kullanilan fonksiyonlar

1)Rampa basamak fonksiyonu r(t). t<0 icin g(t)=0, ve t>0 icin g(t)=t olan fonksiyondur.



Ornek Problem: Birim basamak fonksiyonunun Laplas donusumunu bulun.

Cozum:

$$G(s) = \int_{t=0}^{\infty} g(t)e^{-st}dt = \int_{t=0}^{\infty} u(t)e^{-st}dt = \int_{t=0}^{\infty} e^{-st}dt = \frac{1}{-s}e^{-st}\Big|_{t=0}^{\infty}$$

$$\frac{1}{-s}(e^{-s\infty} - e^{-s0}) = \frac{1}{-s}(e^{-\infty} - e^{0}) = \frac{1}{-s}(0-1) = \frac{1}{s}$$

s>0 varsayildigi icin $e^{-s\infty} = e^{-\infty} = 0$ olacaktir.

Sonuc: ₤{u(t)}= $\frac{1}{s}$,

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Ornek Problem 61: g(t)=e^{at} seklinde verilen fonksiyounun Laplas donusumunu bulun.

Cozum:

$$G(s) = \int_{t=0}^{\infty} g(t)e^{-st}dt = \int_{t=0}^{\infty} e^{at}e^{-st}dt = \int_{t=0}^{\infty} e^{(a-s)t}dt$$

$$\frac{1}{\text{a - s}}e^{(a-s)t}\Big|_{t=0}^{\infty} = \frac{1}{\text{a - s}}(e^{-\infty} - e^{0}) = \frac{1}{\text{s - a}}$$

Sonuc: ₤{ e^{at}}= $\frac{1}{s-a}$,

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Ornek Problem 61: g(t)=t seklinde verilen rampa fonksiyounun Laplas donusumunu bulun.

Cozum

$$G(s) = \int_{t=0}^{\infty} g(t)e^{-st}dt = \int_{t=0}^{\infty} te^{-st}dt$$

Kismi integrasyon kullanilarak

$$\int xe^{ax}dx = \frac{1}{a}xe^{ax} - \frac{1}{a^2}e^{ax}$$

oldugu gosterilebilir. Dolayisiyla

$$G(s) = \int_{t=0}^{\infty} g(t)e^{-st}dt$$

$$= \int_{t=0}^{\infty} te^{-st}dt = \left(\frac{1}{-s}te^{-st} - \frac{1}{(-s)^2}te^{-st}\right)\Bigg|_0^{\infty}$$

$$= \frac{1}{s^2}$$

Sonuc: ₤{ t}= $\frac{1}{s^2}$,

Not: Butun fonksiyonlarin negatif kismi sifir oldugundan Laplas donusumu alinan fonksiyonlar u(t) ile carpilmis olarak gosterilirler.

$£\{ e^{at}\ u(t)\} = \dfrac{1}{s-a}$

$£\{ t\ u(t)\} = \dfrac{1}{s^2}$

Cok Kullanilan fonksiyonlarin Laplas Tablosu

g(t)	G(s)	g(t)	G(s)
u(t)	$\frac{1}{s}$	sin(bt)	$\frac{b}{s^2+b^2}$
t	$\frac{1}{s^2}$	cos(bt)	$\frac{s}{s^2+b^2}$
e^{at}	$\frac{1}{s-a}$	e^{at} sin(bt)	$\frac{b}{(s+a)^2+b^2}$
e^{-at}	$\frac{1}{s+a}$	e^{at} cos(bt)	$\frac{s+a}{(s+a)^2+b^2}$
δ(t)	1		

Ornekler

$£\{ e^{2t} \} = \dfrac{1}{s-2}$

$£\{ e^{-5t} \} = \dfrac{1}{s+5}$

$£\{ e^{-5t}\ \cos(8t)\} = \dfrac{s+5}{(s+5)^2+8^2} = \dfrac{s+5}{s^2+10s+89}$

$£\{ e^{-5t}\ \sin(8t)\} = \dfrac{8}{(s+5)^2+8^2} = \dfrac{8}{s^2+10s+89}$

Laplas Teoremleri.

$\int_0^t f(x)\,dx$	$\frac{F(s)}{s}$
$f(t-a)u(t-a),\ a>0$	$e^{-as}F(s)$
$e^{-at}f(t)$	$F(s+a)$
$f(at),\ a>0$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
$tf(t)$	$-\frac{dF(s)}{ds}$
$t^n f(t)$	$(-1)^n\frac{d^nF(s)}{ds^n}$
$\frac{f(t)}{t}$	$\int_s^\infty F(u)\,du$

Ters Laplas Donusumu

Ters Laplas donusumu tablolar yardimiyla yapilir.

Ornekler

$£^{-1}\left\{\dfrac{1}{s}\right\} = u(t)$

$£^{-1}\left\{\dfrac{1}{s-2}\right\} = e^{2t}$

$£^{-1}\left\{\dfrac{1}{s+12}\right\} = e^{-12t}$

$£^{-1}\left\{\dfrac{2}{s^2+4}\right\} = \sin(2t)$

$£^{-1}\left\{\dfrac{s}{s^2+4}\right\} = \cos(2t)$

$£^{-1}\{1\} = \delta(t)$

Standart forma benzemeyenler benzetilmeye calisilir.

$£^{-1}\left\{\dfrac{15}{s-2}\right\} = 15\, £^{-1}\left\{\dfrac{1}{s-2}\right\} = 15\ e^{2t}$

$£^{-1}\left\{\dfrac{20}{s^2+4}\right\} = 10 £^{-1}\left\{\dfrac{2}{s^2+4}\right\} = 10\sin(2t)$

$£^{-1}\left\{\dfrac{28s}{s^2+4}\right\} = 28 £^{-1}\left\{\dfrac{s}{s^2+4}\right\} = 28\cos(2t)$

$$£^{-1}\left\{\frac{28s+20}{s^2+4}\right\} = £^{-1}\left\{\frac{28s}{s^2+4}+\frac{20}{s^2+4}\right\}$$

$$= £^{-1}\left\{\frac{28s}{s^2+4}\right\} + £^{-1}\left\{\frac{20}{s^2+4}\right\}$$

$$= 28 £^{-1}\left\{\frac{s}{s^2+4}\right\} + 10 £^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= 28\cos(2t) + 10\sin(2t)$$

Bir polinomun derecesi kadar koku vardir.
x+1=0 $\rightarrow$ tek koku var x=-1
x^2-4=0 $\rightarrow$ iki koku var x=-2, x=+2
x^2-5=0 $\rightarrow$ iki koku var x=-2.236, x=2.236
x^2+3x+2=0 $\rightarrow$ iki koku var x=-2, x=-1
x^2 +1=0 $\rightarrow$ iki koku var x=i, x=-i
x^2 +4=0 $\rightarrow$ iki koku var x=2i, x=-2i
x^2+5=0 $\rightarrow$ iki koku var x=-2.236i, x=2.236i
x^2 +2x+1=0 $\rightarrow$ iki koku var x=-1, x=-1
x^2 =0 $\rightarrow$ iki koku var x=0, x=0
x^2 –x=0 $\rightarrow$ iki koku var x=0, x=1
Bir polinomun bir koku kompleks ise onun eslenigi de muhakkak koktur.
x^2 +1=0 $\rightarrow$ x=-i, x=-i
x^2 +2x+2=0 $\rightarrow$ x=-1-i, x=-1+i
x^2 -6x+25=0 $\rightarrow$ x=3+4i, x=3-4i
x^2 +6x+25=0 $\rightarrow$ x=-3+4i, x=-3-4i
x^2 +4x+13=0 $\rightarrow$ x=-2+2i, x=-2-3i
x^2 -4x+13=0 $\rightarrow$ x=2+2i, x=2-3i

Pay ve paydasi polinom olan kesirlere rasyonel kesirler denir. Bir rasyonel kesir paydanin koku kadar basit kesirlere ayrilabilir.

$$\frac{a_n x^n + a_{n-1}x^{n-1} \ldots + a_1 x + a_0}{x^m + b_{m-1}x^{m-1} \ldots + b_1 x + b_0} = \frac{a_n x^n + a_{n-1}x^{n-1} \ldots + a_1 x + a_0}{(x+x_1)(x+x_2)\ldots(x+x_m)}$$

$$= \frac{A_1}{x+x_1} + \frac{A_2}{x+x_2} + \ldots + \frac{B_1}{x+x_5} + \frac{B_2}{(x+x_5)^2} + \frac{B_2}{(x+x_5)^3} + \ldots$$

A ve B lerin toplami m adet
Ornekler

$$\frac{5x+7}{x^2+3x+2} = \frac{2}{x+1} + \frac{3}{x+2},$$

[usler,kok,bolum] = residue ([5 7] , [1 3 2])

$$\frac{5x-7}{x^2-3x+2} = \frac{2}{x-1} + \frac{3}{x-2}$$

[usler,kok,bolum] = residue ([5 -7] , [1 -3 2])

$$\frac{2x+1}{x^2+3x+2} = \frac{3}{x+2} + \frac{-1}{x+1},$$

Kokler kompleks ise carpanlara ayirma isleminde paya gelen terimler de kompleks ve esleniktir.

$$\frac{6x+20}{x^2+4x+5} = \frac{3+4i}{x+2+i} + \frac{3-4i}{x+2-i},$$

$$\frac{6x-4}{x^2-4x+5} = \frac{3+4i}{x-2+i} + \frac{3-4i}{x-2-i}$$

$$\frac{10x-32}{x^2-4x+8}=\frac{5+3i}{x-2-2i}+\frac{5-3i}{x-2+2i}$$

$$\frac{10x-8}{x^2-4x+8}=\frac{5-3i}{x-2-2i}+\frac{5+3i}{x-2+2i}$$

$$\frac{14x^2-74x+72}{x^3-9x^2+28x-40}=\frac{5-3i}{x-2-2i}+\frac{5+3i}{x-2+2i}+\frac{4}{x-5}$$

[usler,kok,bolum] = residue ([14 -74 72] , [1 -9 28 -40])

$$\frac{14x^2+26x-8}{x^3+x^2-12x+40}=\frac{5-3i}{x-2-2i}+\frac{5+3i}{x-2+2i}+\frac{4}{x+5}$$

$$\frac{16x^3+42x^2-6x+72}{x^4+2x^3-11x^2+28x+40}=\frac{5-3i}{x-2-2i}+\frac{5+3i}{x-2+2i}+\frac{4}{x+5}+\frac{2}{x+1}$$

$$\frac{12x^3-106x^2+434x-440}{x^4-10x^3+57x^2-148x+200}=\frac{5-3i}{x-2-2i}+\frac{5+3i}{x-2+2i}+\frac{1+3i}{x-3-4i}+\frac{1-3i}{x-3+4i}$$

$$=\frac{10x-8}{x^2-4x+8}+\frac{2x-30}{x^2-6x+25}$$

$$\frac{3x+1}{x^2-2x+1}=\frac{3}{x-1}+\frac{4}{(x-1)^2},$$

[usler,kok,bolum] = residue ([3 1] , [1 -2 1])

$$\frac{7x+9}{x^2+2x+1}=\frac{7}{x+1}+\frac{2}{(x+1)^2}$$

$$\frac{10x}{x^2+2x+1}=\frac{10}{x+1}+\frac{-10}{(x+1)^2}$$

$$\frac{17x^5-282x^3-153x^2+1172x+344}{x^6-4x^5-15x^4+50x^3+100x^2-168x-288}=\frac{1}{x+2}+\frac{5}{(x+2)^2}+\frac{6}{(x+2)^3}$$

$$+\frac{7}{x-3}+\frac{8}{(x-3)^2}+\frac{9}{(x-4)}$$

Basit Kesırlere Ayırma .

$$f(z)=\frac{p(z)}{q(z)}=\frac{A}{z-a}+\frac{B}{z-b}+\frac{C}{z-c}.+.......$$

I. **rezidu Formulu**

$$A=\operatorname{Res}(z=a)=\lim_{z\to a}(z-a)\frac{p(z)}{q(z)}$$

$$B=\operatorname{Res}(z=b)=\lim_{z\to b}(z-b)\frac{p(z)}{q(z)}$$

$$C=\operatorname{Res}(z=c)=\lim_{z\to c}(z-c)\frac{p(z)}{q(z)}$$

2. **rezidu Formulu**

$$A=\operatorname{Res}(z=a)=\lim_{z\to a}\frac{p(z)}{q'(z)}$$

$$B=\operatorname{Res}(z=b)=\lim_{z\to b}\frac{p(z)}{q'(z)}$$

$$C=\operatorname{Res}(z=c)=\lim_{z\to c}\frac{p(z)}{q'(z)}$$

Örnek AE-611

$$\frac{z+5}{z^3-5z^2-2z+24}=\frac{z+5}{(z-4)(z+2)(z-3)}$$

$$=\frac{A}{z-4}+\frac{B}{z+2}+\frac{C}{z-3}$$

$$A=\lim_{z\to 4}(z-4)\frac{p(z)}{q(z)}=\lim_{z\to 4}(z-4)\frac{z+5}{(z-4)(z-3)(z+2)}$$

$$=\lim_{z\to 4}\frac{z+5}{(z-3)(z+2)}=\frac{4+5}{(4-3)(4+2)}=\frac{9}{6}=1.5$$

$$B=\lim_{z\to -2}(z-(-2))\frac{p(z)}{q(z)}=\lim_{z\to -2}(z+2)\frac{z+5}{(z-4)(z-3)(z+2)}$$

$$=\lim_{z\to -2}\frac{z+5}{(z-3)(z-4)}=\frac{-2+5}{(-2-3)(-2-4)}=\frac{3}{30}=0.1$$

$$C=\lim_{z\to 3}(z-3)\frac{p(z)}{q(z)}=\lim_{z\to 3}\frac{z+5}{(z-4)(z+2)}=\frac{3+5}{(3-4)(3+2)}=-1.6$$

2. **rezidu Formulunu kullanarak**

p(z)=z+5 *q(z)*= $z^3-5z^2-2z+24$ and

q'(z)= $3z^2-10z-2$

$$A=\lim_{z\to 4}\frac{z+5}{3z^2-10z-2}=\frac{4+5}{3\ 4^2-10\ \ 4-2}=\frac{9}{6}=1.5$$

$$B=\lim_{z\to -2}\frac{z+5}{3z^2-10z-2}=\frac{-2+5}{3\ (-2)^2-10\ \ (-2)-2}=\frac{3}{30}=0.1$$

$$C=\lim_{z\to 3}\frac{z+5}{3z^2-10z-2}=\frac{3+5}{3\ \ (3)^2-10\ (3)-2}=\frac{8}{-5}=-1.6$$

Klasik method

$$\frac{z+5}{z^3-5z^2-2z+24}=\frac{A}{z-4}+\frac{B}{z+2}+\frac{C}{z-3}$$

$$=\frac{A(z+2)(z-3)+B(z-4)(z-3)+C(z-4)(z+2)}{(z-4)(z+2)(z-3)}$$

$$=\frac{A(z^2-z-6)+B(z^2-7z-12)+C(z^2-2z-8)}{(z-4)(z+2)(z-3)}$$

$$=\frac{z^2(A+B+C)+z(-7A-B-2C)+12A-6B-8C}{(z-4)(z+2)(z-3)}$$

$$z^2(A+B+C)+z(-7A-B-2C)+12A-6B-8C=z+5$$

$$A+B+C=0,\qquad -7A-B-2C=1,\qquad 12A-6B-8C=5$$

Solving for A,B,C we get A=1.5 B=0.1 C=-1.6

$$\frac{z+5}{z^3-5z^2-2z+24}=\frac{1.5}{z-4}+\frac{0.1}{z+2}+\frac{-1.6}{z-3}$$

Örnek AE-612

$$\frac{2z+12}{z^2+2z+2}=\frac{2z+12}{[z-(-1+i)][z-(-1-i)]}=\frac{2z+12}{(z+1-i)(z+1+i)}$$

$$=\frac{A}{(z+1-i)}+\frac{B}{(z+1+i)}$$

first residue formula

$$A=\lim_{z\to -1+i}(z+1-i)\frac{2z+12}{(z+1-i)(z+1+i)}=\lim_{z\to -1+i}\frac{2z+12}{(z+1+i)}$$

$$=\frac{2(-1+i)+12}{((-1+i)+1+i)}=\frac{2i+10}{2i}=\frac{2i}{2i}+\frac{10}{2i}=1-5i$$

$$B=\lim_{z\to -1-i}(z+1+i)\frac{2z+12}{(z+1-i)(z+1+i)}=\lim_{z\to -1-i}\frac{2z+12}{(z+1-i)}=1+5i$$

second residue formula

p(z)=2z+12, q(z)=z^2+2z+2, *q'*(z)=2z+2

$$A=\lim_{z\to -1+i}\frac{p(z)}{q'(z)}=\frac{2z+12}{2z+2}=\frac{2(-1+i)+12}{2(-1+i)+2}=1-5i$$

$$B=\lim_{z\to -1-i}\frac{p(z)}{q'(z)}=\frac{2z+12}{2z+2}=\frac{2(-1-i)+12}{2(-1-i)+2}=1+5i$$

classical Method

$$\frac{2z+12}{z^2+2z+2}=\frac{A}{(z+1-i)}+\frac{B}{(z+1+i)}=\frac{A(z+1+i)+B(z+1-i)}{(z+1-i)(z+1+i)}$$

(A+B)=2 A(1+i)+B(1-i)=12 solution is A=1-5i , B=1+5i

Sonuc $\dfrac{2z+12}{z^2+2z+2}=\dfrac{1+5i}{(z+1-i)}+\dfrac{1-5i}{(z+1+i)}$

Diferansiyel Denklemin Genel cozumunu Laplas Donusumu ile bulma:

₤{y}=Y(s)

₤{ $\frac{dy}{dt}$ }=₤{y'}=sY(s)-y(0)

₤{ $\frac{d^2y}{dt^2}$ }=₤{ y''}=s^{2}Y(s)- s y(0) - y'(0)

₤{ $\frac{d^3y}{dt^3}$ }=₤{ y'''}=s^{3}Y(s)- s^{2}y(0) – s y'(0) - y''(0)

baslangic kosullari sifir ise y(0)=0, y'(0)=0, y''(0)=0 denklemler basitlesir

₤{y}=Y(s)

₤{y'}=sY(s)

₤{ y''}=s^{2}Y(s)

₤{y'''}=s^{3}Y(s)

Ornek Problem 271) y' +2y =0, diff denkleminin baslangic kosullari y(0)=5 olarak veriliyor. Diff denklemin genel cozumunu bulun.

Cozum:

₤{y}=Y(s)

₤{ $\frac{dy}{dt}$ }=₤{y'}= sY(s)-y(0) = sY(s) - 5

y' +2y =0, ==> s Y(s) – 5+2 Y(s)=0

Y(s) [s +2] = 5

$Y(s)=\frac{5}{s+2}$ **==> y(t)=5 e^{-2t}** ,

Ornek Problem 272) y' +2y =2u(t), diff denkleminin baslangic kosullari y(0)=5 Diff denklemin genel cozumunu bulun.

Cozum:

₤{y}=Y(s)

₤{ $\frac{dy}{dt}$ }=₤{y'}= sY(s)-y(0) = sY(s) - 5

₤{ u(t) }= $\frac{1}{s}$

y' +2y =2u(t), ==> s Y(s) – 5+2 Y(s)= $2\frac{1}{s}$

Y(s) [s +2] - 5= $\frac{2}{s}$

Y(s) [s +2] = 5+ $\frac{2}{s}$

Y(s) [s +2] = $\frac{5s+2}{s}$

$Y(s) = \frac{5s+2}{s(s+2)}$

$\frac{A}{s}+\frac{B}{s+2}=\frac{A(s+2)+Bs}{s(s+2)}$

A+B=5,

2A=2

A=1, B=4

$Y(s)=\frac{1}{s}+\frac{4}{s+2}$

y(t)= u(t) + 4 e^{-2t}

Ornek Problem 273) y' +2y =8sin(6t) , diff denkleminin baslangic kosullari y(0)=0, Diff denklemin genel cozumunu bulun.

Cozum:

₤{y}=Y(s)

₤{ $\frac{dy}{dt}$ }=₤{y'}= sY(s)-y(0) = sY(s)-0

₤{ sin(5t) }= $\frac{6}{s^2+36}$

y' +2y =8sin(6t) , ==>

s Y(s) +2 Y(s)=8 $\frac{6}{s^2+36}$

Y(s) [s +2] = $\frac{48}{s^2+36}$

$Y(s) = \frac{48}{(s+2)(s^2+36)}$

$$\frac{48}{(s+2)(s^2+36)}=\frac{A}{(s+2)}+\frac{Bs+C}{s^2+36}$$

$$=\frac{A(s^2+36)+(Bs+C)(s+2)}{(s+2)(s^2+36)}$$

$$=\frac{(A+B)s^2+(2B+C)s+36A+2C}{(s+2)(s^2+36)}$$

A+B=0, 2B+C=0, 36A+2C=48

inv([1 1 0; 0 2 1; 36 0 2])*[0; 0; 48]

A=1.2, B=-1.2, C=2.4

$$Y(s)=\frac{1.2}{(s+2)}+\frac{-1.2s+2.4}{s^2+36}$$

y(t)=1.2 e^{-2t} + **?**

$\frac{-1.2s+2.4}{s^2+36}$ nin Laplas donusumu nedir.

$$\frac{-1.2s+2.4}{s^2+36} = \frac{-1.2(s-2)}{s^2+36} = -1.2\frac{(s-2)}{s^2+36}$$
$$= -1.2\left(\frac{s}{s^2+36} - \frac{2}{s^2+36}\right)$$
$$= -1.2\left(\frac{s}{s^2+36} - \frac{3}{3}\frac{2}{s^2+36}\right)$$
$$= -1.2\left(\frac{s}{s^2+36} - \frac{1}{3}\frac{6}{s^2+36}\right)$$

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+36}\right) = \cos(6t)$$
$$\mathcal{L}^{-1}\left(\frac{6}{s^2+36}\right) = \sin(6t)$$

oldugundan , $\frac{-1.2s+2.4}{s^2+36}$ ifadesinin Laplas donusumu

$$-1.2(\cos(6t) - \frac{1}{3}\sin(6t))$$
$$-1.2\cos(6t) - \frac{1.2}{3}\sin(6t)$$

elde edilir.

Bu ifade y(t) de yerine konuluirsa

$$y(t)=1.2e^{-2t} + -1.2\cos(6t) - \frac{1.2}{3}\sin(6t)$$

elde edilr.

Ornek Problem 275) $\mathbf{y'' - 3y' + 2y = 4t^2}$ diff denkleminin baslangic kosullari y(0)=0 ve y'(0)=0 olarak veriliyor. Diff denklemin genel cozumunu bulun.

$\mathbf{y'' - 3y' + 2y = 4t^2}$

$\mathcal{L}\{\mathbf{y'' - 3y' + 2y}\} = \mathcal{L}\{\mathbf{4t^2}\}$

$\mathcal{L}\{\mathbf{y''}\} - \mathcal{L}\{\mathbf{3y'}\} + \mathcal{L}\{\mathbf{2y}\} = \mathcal{L}\{\mathbf{4t^2}\}$

$$s^2Y(s) - 3sY(s) + 2Y(s) = 4\frac{2}{s^3} = \frac{8}{s^3}$$

$$(s^2 - 3s + 2)Y(s) = \frac{8}{s^3}$$

$$Y(s) = \frac{8}{s^3(s^2-3s+2)}$$

$$Y(s) = \frac{8}{s^3(s^2-3s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s-2} + \frac{E}{s-1}$$

$$\underset{S^2(S-2)(S-1)}{\frac{A}{s}} + \underset{S(S-2)(S-1)}{\frac{B}{s^2}} + \underset{(S-2)(S-1)}{\frac{C}{s^3}} + \underset{S^3(S-1)}{\frac{D}{s-2}} + \underset{S^3(S-2)}{\frac{E}{s-1}}$$

$$= \frac{As^2(s-2)(s-1) + Bs(s-2)(s-1)) + C(s-2)(s-1)) + Ds^3(s-1) + Es^3(s-2)}{s^3(s-2)(s-1)}$$

$$= \frac{A(s^4 - 3s^3 + 2s^2) + B(s^3 - 3s^2 + 2s) + C(s^2 - 3s + 2) + D(s^4 - s^3) + E(s^4 - 2s^3)}{s^3(s-2)(s-1)}$$

$$= \frac{(A+D+E)s^4 + (-3A+B-D-2E)s^3 + (2A-3B+C)s^2 + (2B-3C)s + 2C}{s^3(s-2)(s-1)}$$

A+D+E=0

-3A+B-D-2E=0

2A-3B+C=0

2B-3C=0

2C=8,
A,B,C,D,E bilinen yontemlerle hesaplanir.
A=7, B=6, C=4, D=1, E= -8

$$Y(s)=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{D}{s-2}+\frac{E}{s-1}=\frac{7}{s}+\frac{6}{s^2}+\frac{4}{s^3}+\frac{1}{s-2}+\frac{-8}{s-1}$$

$y(x)= 7+6t + 4\,\frac{t^2}{2} + e^{2t} - 8\,e^t = 7+6t + 2t^2 + e^{2t} - 8\,e^t$

****

qq11 $\mathcal{L}[x(t)] = X(s)$ ise $\mathcal{L}[x(t)e^{at}] = X(s-a)$ oldugunu gosteriniz. (??.ozellik)

Cozum:

$$\mathcal{L}[x(t)] = \int_0^{\infty} e^{-st}x(t)dt = X(s)$$

oldugundan

$$\mathcal{L}[x(t)e^{at}] = \int_0^{\infty} e^{-st}x(t)e^{at}dt = \int_0^{\infty} e^{-(s-a)t}x(t)dt = X(s-a)$$

olacagi aciktir.

qq12 $\mathcal{L}[x(t)] = X(s)$ ise $\mathcal{L}[tx(t)] = -\frac{dX(s)}{ds}$ oldugunu gosteriniz.

Cozum:

$$X(s) = \mathcal{L}[x(t)] = \int_0^{\infty} e^{-st}x(t)dt$$

oldugundan

$$\frac{dX(s)}{ds} = \frac{d}{ds}\int_0^{\infty} e^{-st}x(t)dt = \int_0^{\infty} \frac{d}{ds}(e^{-st}x(t))dt$$

$$= \int_0^{\infty} -te^{-st}x(t)dt = -\int_0^{\infty} e^{-st}(tx(t))dt = -\mathcal{L}[tx(t)]$$

olacaktir.

qq13 $\mathcal{L}[x(t)] = X(s)$ ise $\mathcal{L}\left[\frac{dx(t)}{dt}\right] = sX(s) - x(0)$ oldugunu gosteriniz.

Cozum:

$$\mathcal{L}\left[\frac{dx(t)}{dt}\right] = \int_0^{\infty} e^{-st}\frac{dx(t)}{dt}dt = \lim_{P\infty}\int_0^{P} e^{-st}\frac{dx(t)}{dt}dt$$

$$u = e^{-st},\ \ du = -se^{-st},\quad dv = \frac{dx(t)}{dt}dt,\quad v = x(t)$$

tanimi yapip kismi integrasyon uygulanirsa

$$\mathcal{L}\left[\frac{dx(t)}{dt}\right] == \lim_{P\infty}\left(e^{-st}x(t)|_0^P - (-s)\int_0^{P} e^{-st}x(t)dt\right)$$

$$= \lim_{P\infty}\left(e^{-sP}x(P) - x(0) + s\int_0^{P} e^{-st}x(t)dt\right)$$

$e^{-\infty} = 0$ oldugundan Laplas donusumu alinabilen bir fonksiyon icin $\lim_{P\infty} e^{-sP}x(P) = 0$ olmak zorundadir. Dolayisiyla

$$\mathcal{L}\left[\frac{dx(t)}{dt}\right] = s\int_0^\infty e^{-st}x(t)dt - x(0) = sx(s) - x(0)$$

olarak elde edilir.

qs50 $x_1(t) = e^{-3t}\sin(5t), \quad x_2(t) = e^{-3t}\cos(5t), \quad x_3(t) = e^{-3t}(10\sin(5t) + 20\cos(5t))$ ifadelerinin Laplas donusumlerini bulun.

Cozum: Laplas donusum tablosundan

$$\mathcal{L}[\sin(5t)] = \frac{5}{s^2+25} \qquad \mathcal{L}[\cos(5t)] = \frac{s}{s^2+25}$$

olarak bulunur. (C.P.ref: xq7pc11)'den

$$\mathcal{L}[e^{-3t}\sin(5t)] = \frac{5}{(s+3)^2+25} = \frac{5}{s^2+6s+34}$$

$$\mathcal{L}[e^{-3t}\cos(5t)] = \frac{s+3}{(s+3)^2+25} = \frac{s+3}{s^2+6s+34}$$

olarak bulunur. Laplas donusumunun lineerlik ozelligi kullanilarak

$$\mathcal{L}[e^{-3t}(10\sin(5t)+20\cos(5t))] = 10\frac{5}{s^2+6s+34} + 20\frac{s+3}{s^2+6s+34} = \frac{20s+110}{s^2+6s+34}$$

bulunur.

qs51 $x(t) = t^2e^{8t}$ olduguna gore $X(s)$'yi hesaplayin.

Cozum: Tablodan $Z(s) = \mathcal{L}[t^2] = \frac{2}{s^3}$ oldugu bulunarak,

$$X(s) = \mathcal{L}[t^2e^{8t}] = Z(s-8) = \frac{2}{(s-8)^3}$$

seklinde olacagi acikca gorulur.

$\mathcal{L}[x(t)] = X(s)$ olduguna gore $\mathcal{L}[x(at)] = \frac{1}{a}X(\frac{s}{a})$ oldugunu gosterin.

Cozum:

$$\mathcal{L}[x(at)] = \int_0^\infty e^{-st}x(at)dt$$

$t = u/a, \quad dt = du/a$ donusumu yaparak

$$\begin{aligned}\mathcal{L}[x(at)] &= \int_0^\infty e^{-s(u/a)}x(u)\frac{du}{a} \\ &= \frac{1}{a}\int_0^\infty e^{-su/a}x(u)du \\ &= \frac{1}{a}X\left(\frac{s}{a}\right)\end{aligned}$$

qs53 $\mathcal{L}[t\sin(at)]$'yi hesaplayin.

Cozum: $\mathcal{L}[\sin(at)] = \frac{a}{s^2+a^2}$ ve $\mathcal{L}[tx(t)] = -\frac{dX(s)}{ds}$ oldugundan

$$[t\sin(at)] = -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = \frac{2as}{s^2+a^2}$$

olarak eldeedilir.

$X(s) = \frac{4s-16}{s^2-2s+5}$ ise $x(t)$ nedir.

Cozum: Payda polinomunun kokleri hesaplanirsa $s_1 = 1+2j, \quad s_2 = 1-2j$ olarak bulunur. O halde

$$X(s) = \frac{4s-16}{s^2-2s+5} = \frac{A}{s-(1+2j)} + \frac{B}{s-(1-2j)}$$

seklinde carpanlara ayrilabilir. A ve B katsayilari (Ek-ref: appx41) de gosterilen yontemlerle hesaplanirsa

$$A = 2+3j, \quad B = 2-3j$$

olarak bulunur. O halde

$$X(s) = \frac{4s-16}{s^2-2s+5} = \frac{2+3j}{s-(1+2j)} + \frac{2-3j}{s-(1-2j)}$$

olacaktir. Her terimin ayri ayri ters Laplas donusumu alinirsa

$$x(t) = \mathcal{L}^{-1}[X(S)] = \mathcal{L}^{-1}\left[\frac{2+3j}{s-(1+2j)}\right] + \mathcal{L}^{-1}\left[\frac{2-3j}{s-(1-2j)}\right]$$

$$\begin{aligned}
x(t) &= (2+3j)e^{(1+2j)t} + (2-3j)e^{(1-2j)t} \\
&= (2+3j)e^{t}e^{2jt} + (2-3j)e^{t}e^{-2jt} \\
&= e^{t}((2+3j)e^{2jt} + (2-3j)e^{-2jt}) \\
&= e^{t}(2e^{2jt} + 2e^{-2jt} + 3je^{2jt} - 3je^{-2jt}) \\
&= e^{t}(2(e^{2jt} + e^{-2jt}) + 3j(e^{2jt} - e^{-2jt})) \\
&= e^{t}\left(4\frac{e^{2jt}+e^{-2jt}}{2} + 3j\,2j\frac{e^{2jt}-e^{-2jt}}{2j}\right) \\
&= e^{t}(4\cos(2t) - 6\sin(2t))
\end{aligned}$$

olarak bulunur.

Simdi problemi baska bir yontemle cozelim. Payda polinomunun kokleri kompleks oldugundan $x(t)$'nin sinuzoidal terimler ihtiva edecegi aciktir.

$$\mathcal{L}^{-1}\left[\frac{a}{(s+a)^2+b^2}\right] = e^{-at}\sin(bt) \quad \mathcal{L}^{-1}\left[\frac{s+a}{(s+a)^2+b^2}\right] = e^{-at}\cos(bt)$$

oldugundan verilen ifadeyi bu formlara benzetmeye calisalim. $X(s)$'nin paydasinin yukaridaki forma benzemesi icin

$$(s+a)^2 + b^2 = s^2 + 2as + a^2 + b^2 = s^2 - 2s + 5$$

olmalidir. Buradan acikca gorulecegi gibi

$$2a = -2 \ \rightarrow a = -1, \quad ve \quad a^2 + b^2 = 5 \ \rightarrow b = 2$$

olmalidir. $x(t)$ ifadesi

$$\mathcal{L}^{-1}\left[\frac{-1}{(s-1)^2 + 2^2}\right] = e^t \sin(2t) \quad \mathcal{L}^{-1}\left[\frac{s-1}{(s-1)^2 + 2^2}\right] = e^t \cos(2t)$$

terimlerini icinde bulunduracaktir. O halde $X(s)$ ifadesini yukaridaki bilesenler cinsinden yazmak gerekir.

$$X(s) = \frac{4s-16}{s^2-2s+5} = \frac{4s-4-12}{s^2-2s+5} = 4\frac{(s-1)}{(s-1)^2+2^2} - 6\frac{2}{(s-1)^2+2^2}$$

$X(s)$'nin ters Laplas donusumu alinirsa

$$x(t) = e^t(4\cos(2t) - 6\sin(2t))$$

olarak bulunur.

qq21 $x(s) = \frac{s}{s+a}$ ise $x(t) = ?$

Cozum: $\mathcal{L}[e^{-at} = \frac{1}{s+a}$ ve $s(X(s) - x(0) = \frac{d}{dt}x(t)$ bagintilari kullanilarak

$$\mathcal{L}^{-1}\left[s\left(\frac{1}{s+a}\right)\right] - e^{-a\,0} = \frac{d}{dt}e^{-at}$$

$$\mathcal{L}^{-1}\left[s\frac{1}{s+a}\right] = \frac{d}{dt}e^{-at} + 1$$

$$\mathcal{L}^{-1}\left[\frac{s}{s+a}\right] = -ae^{-at} + 1$$

olarak bulunur.

qq14 $X(s) = \frac{12s^3-14s^2+152s-294}{s^4-6s^3+42s^2-78s+145}$ ise $x(t)$ nedir.

Cozum: Payda polinomunun kokleri hesaplanirsa

$$s_1 = 1+2j, \quad s_2 = 1-2j, \quad s_3 = 2+5j, \quad s_4 = 2-5j$$

olarak bulunur. O halde

$$X(s) = \frac{12s^3 - 14s^2 + 152s - 294}{s^4 - 6s^3 + 42s^2 - 78s + 145}$$

$$= \frac{A}{s-(1+2j)} + \frac{B}{s-(1-2j)} + \frac{C}{s-(2+5j)} + \frac{D}{s-(2-5j)}$$

seklinde carpanlara ayrilabilir. A,B,C,D katsayilari hesaplanirsa

$$A = 2+3j, \quad B = 2-3j, \quad C = 4-5j, \quad ,D = 4+5j$$

olarak bulunur. Bu adimdan sonraki kisim (C.P.ref: xq7pc153)'de oldugu gibi iki degisik yontemle yapilabilir.

Birinci yontemde (C.P.ref: xq7pc153)'de oldugu gibi A,B,C,D katsayilari yerine konur ve her terimin ayri ayri ters Laplas donusumu alinir.

$$X(s) = \frac{2+3j}{s-(1+2j)} + \frac{2-3j}{s-(1-2j)} + \frac{4-5j}{s-(2+5j)} + \frac{4+5j}{s-(2-5j)}$$

$$x(t) = (2+3j)e^{(1+2j)t} + (2-3j)e^{(1-2j)t} + (4-5j)e^{2+5j)t} + (4+5j)e^{2-5j)t}$$

Daha sonra reel ve sanal kisimlar uygun sekilde guruplandirilarak sinuslu ve kosinuslu terimler elde edilir.

$$x(t) = e^{t}(4\cos(2t) - 6\sin(2t)) \ + e^{2t}(8\cos(5t) + 10\sin(5t))$$

olarak bulunur

$X(s)$'nin payi ve paydasi reel katsayili oldugundan elde edilecek $x(t)$'deki kompleks degisken j carpani daima yok edilir ve $x(t)$ hicbir zaman j carpani bulundurmaz.

Ikinci yontemde kompleks eslenik kokler birlestirilerek $X(s)$ iki terim gibi dusunulur.

$$X(s) = \left(\frac{2+3j}{s-(1+2j)} + \frac{2-3j}{s-(1-2j)}\right) + \left(\frac{4-5j}{s-(2+5j)} + \frac{4+5j}{s-(2-5j)}\right)$$

$$= \frac{4s-16}{s^2-2s+5} + \frac{8s+34}{s^2-4s+29}$$

Tipki (C.P.ref: xq7pc153)'de oldugu gibi iki terimin ters Laplas donusumleri alinir. birinci terim (C.P.ref: xq7pc153)'de oldugu gibi Ikinci terim

$$8\frac{s-2}{(s-2)^2+5^2} + 10\frac{5}{(s-2)^2+5^2}$$

haline getirilir. Sonucta $x(t)$ fonksiyonu

$$x(t) = e^{t}(4\cos(2t) - 6\sin(2t)) \ + e^{2t}(8\cos(5t) + 10\sin(5t))$$

olarak elde edilir.

qq15 $X(s) = \frac{20s^2+5s-2}{s^3(s+2)}$ ise $x(t)$ nedir.

Cozum: Payda polinomunun kokleri $s_1 = -1, \quad s_2 = 0, \quad s_3 = 0, \quad s_4 = 0$ seklindedir. Yani $s = 0$ da 3 katli kok vardir. O halde $X(s)$ ifadesi

$$X(s) = \frac{20s^2+5s-2}{s^3(s+2)} = \frac{A}{s+2} + \frac{B}{s} + \frac{C}{s^2} + \frac{D}{s^3}$$

seklinde basit kesirler halinde yazilabilir. A,B,C,D katsayilari hesaplanirsa

$$A = -9, \quad ,B = 9 \quad ,C = 2, \quad D = 1$$

olarak bulunur. O halde $X(s)$ fonksiyonu

$$X(s) = \frac{-9}{s+2} + \frac{9}{s} + \frac{2}{s^2} + \frac{1}{s^3}$$

seklinde olacaktir. Laplas donusumleri tablosuna bakarak

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = u(t), \quad \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t \quad \mathcal{L}^{-1}\left[\frac{1}{s^3}\right] = \frac{t^2}{2}$$

oldugu gozonune alinip $X(s)$nin ters Laplas donusumu alinirsa

$$x(t) = -9e^{2t} + 9u(t) + 2t + \frac{1}{2}t^2$$

elde edilir.

qq16 $X(s) = \frac{7s^4+36s^3+41s^2-2s-1}{s^5+4s^4+s^3-10s^2-4s+8}$ ise $x(t)$ nedir.

Cozum: Onceki problemlere benzer sekilde $X(s)$ fonksiyonu

$$X(s) = \frac{7s^4 + 36s^3 + 41s^2 - 2s - 1}{s^5 + 4s^4 + s^3 - 10s^2 - 4s + 8}$$

$$= \frac{2}{s+2} + \frac{4}{(s+2)^2} + \frac{-1}{(s+2)^3} + \frac{5}{s-1} + \frac{3}{(s-1)^2}$$

seklinde yazilabilir. $\frac{1}{s}$, $\frac{1}{s^2}$, $\frac{1}{s^3}$ fonksiyonlarinin ters Laplas donusumleri (C.P.ref: xq7pc171)de verilmisti.

$$\mathcal{L}[e^{at}x(t)] = X(s-a)$$

oldugu gozoune alinirsa

$$\mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right] = te^t \quad \mathcal{L}^{-1}\left[\frac{1}{(s+2)^2}\right] = te^{-2t} \quad \mathcal{L}^{-1}\left[\frac{1}{(s+2)^3}\right] = \frac{1}{2}t^2e^{-2t}$$

olarak bulunur. Sonuc olarak $x(t)$ fonksiyonu

$$x(t) = 2e^{-2t} + 4te^{-2t} + -\frac{1}{2}t^2e^{-2t} + 5e^t + 3te^t$$

seklinde elde edilir.

qq17 $X(s) = \frac{4s^3-16s^2-4s-64}{s^4-4s^3+14s^2-20s+25}$ ise $x(t)$ nedir.-

Cozum: Payda polinomunun kokleri hesaplanirsa

$$s_1 = 1+2j, \quad s_2 = 1-2j \quad s_3 = 1+2j, \quad s_4 = 1-2j$$

olarak bulunur. O halde

$$X(s) = \frac{4s^3 - 16s^2 - 4s - 64}{s^4 - 4s^3 + 14s^2 - 20s + 25}$$

$$= \frac{A}{s-(1+2j)} + \frac{A^*}{s-(1-2j)} \frac{B}{[s-(1+2j)]^2} + \frac{B^*}{[s-(1-2j)]^2}$$

seklinde carpanlara ayrilabilir. A ve B katsayilari (Ek-ref: appx41) de gosterilen yontemlerle hesaplanirsa

$$A = 2+3j, \quad B = 4+5j;$$

olarak bulunur ve $X(s)$ ifadesi

$$X(s) = \frac{2+3j}{s-(1+2j)} + \frac{2-3j}{s-(1-2j)} \frac{4+5j}{[s-(1+2j)]^2} + \frac{4-5j}{[s-(1-2j)]^2}$$

seklinde olacaktir. Ilk iki terimin ters Laplas donusumu (C.P.ref: xq7pc153)de oldugu gibi hesaplanir Son iki terimin ters Laplas donusumu inverse Laplace Transform

1	$\delta(t)$
$\frac{1}{s}$	$u(t)$
$\frac{a}{s^2+a^2}$	$\sin at$
$\frac{s}{s^2+a^2}$	$\cos at$
$\frac{1}{s+a}$	e^{-at}

Example problem: Calculate inverse Laplace Trans

a) $F_1(s) = \frac{4}{s}$ b) $F_2(s) = 4 + \frac{5}{s}$ c) $F_3(s) = \frac{1}{s+6}$

d) $F_4(s) = \frac{8}{s+10}$ e) $F_5(s) = \frac{3}{s} + \frac{4}{s-2} - \frac{6}{s+3}$ f) $F_6(s) = \frac{1}{s^2+4}$

Solution

a) $F_1(s) = 4 \cdot \frac{1}{s} \Longrightarrow f_1(t) = 4\, u(t)$

b) $F_2(s) = 4 + \frac{5}{s} \Longrightarrow f_2(t) = 4\,\delta(t) + 5\,u(t)$

c) $F_3(s) = \frac{1}{s+6} \Longrightarrow f_3(t) = e^{-6t}$

d) $F_4(s) = \frac{8}{s+10} = 8\,\frac{1}{s+10} \Rightarrow f_4(t) = 8\,e^{-10t}$

e) $F_5(s) = \frac{3}{s} + 4\,\frac{1}{s-2} - 6\,\frac{1}{s+3} \Rightarrow f_5(t) = 3\,u(t) + 4\,e^{2t} - 6e^{-3t}$

f) $F_6(s) = \frac{1}{s^2+4} = \frac{1}{2}\,\frac{2}{s^2+2^2} \Rightarrow \frac{1}{2}\,\sin 2t$

Example problem: calculate $f(t)$

a) $F_1(s) = \frac{2}{s} + \frac{3}{s+1}$ b) $f_2(s) = \frac{3}{s+2} + \frac{4}{s^2+9}$

solution

a) $f_1(s) = 2 \cdot \frac{1}{s} + 3 \frac{1}{s+1} \Rightarrow f_1(t) = 2\,u(t) + 3\,e^{-t}$

b) $f_2(s) = 3 \frac{1}{s+2} + 4 \frac{1}{3} \frac{3}{s^2+3^2}$

$$f_2(t) = 3\,e^{-2t} + \frac{4}{3} \sin 3t$$

Example problem calculate $f(t)$

$$f_1(s) = \frac{10s}{s^2+9} + \frac{8}{s^2+9} = 10 \frac{s}{s^2+3^2} + \frac{8}{3} \frac{3}{s^2+3^2}$$

$$f_1(t) = 10 \cos 3t + \frac{8}{3} \sin 3t$$

Example problem calculate $f(t)$

a) $$f(s) = \frac{5s+4}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2} = \frac{A(s+2) + Bs}{s(s+2)} = \frac{(A+B)s + 2A}{s(s+2)}$$

$$\frac{5s+4}{s(s+2)} = \frac{(A+B)s + 2A}{s(s+2)}$$

$$\begin{pmatrix} 5 = A+B \\ 4 = 2A \end{pmatrix} \Rightarrow \begin{matrix} A = 2 \\ B = 3 \end{matrix}$$

$$F(s) = \frac{5s+4}{s(s+2)} = \frac{2}{s} + \frac{3}{s+2}$$

$$f(t) = 2\,u(t) + 3\,e^{-2t}$$

Example problem calculate $f(t)$

$$F(s) = \frac{9s^2+52s+67}{s^3+9s^2+23s+15}$$

$$s^3+9s^2+23s+15=0 \Rightarrow \begin{array}{l} s_1=-1 \\ s_2=-3 \\ s_3=-5 \end{array}$$

$$\frac{9s^2+52s+67}{s^3+9s^2+23s+15} = \frac{9s^2+52s+67}{(s+1)(s+3)(s+5)} = \frac{A}{s+1} + \frac{B}{s+3} + \frac{C}{s+5}$$

calculate A, B, C

$A=2$
$B=3$
$C=4$

$$F(s) = \frac{9s^2+52s+67}{s^3+9s^2+23s+15} = \frac{2}{s+1} + \frac{3}{s+3} + \frac{4}{s+5}$$

$$f(t) = 2e^{-t} + 3e^{-3t} + 4e^{-5t}$$

Problem: How can we calculate A, B, C.

Residues: Partial Fraction Expansion.

$$f(z)=\frac{p(z)}{q(z)}=\frac{A}{z-a}+\frac{B}{z-b}+\frac{C}{z-c}+\ldots\ldots$$

First residue Formula

$$A=\operatorname{Res}(z=a)=\lim_{z\to a}(z-a)\frac{p(z)}{q(z)}$$

$$B=\operatorname{Res}(z=b)=\lim_{z\to b}(z-b)\frac{p(z)}{q(z)}$$

$$C=\operatorname{Res}(z=c)=\lim_{z\to c}(z-c)\frac{p(z)}{q(z)}$$

Second residue Formula

$$A=\operatorname{Res}(z=a)=\lim_{z\to a}\frac{p(z)}{q'(z)}$$

$$B=\operatorname{Res}(z=b)=\lim_{z\to b}\frac{p(z)}{q'(z)}$$

$$C=\operatorname{Res}(z=c)=\lim_{z\to c}\frac{p(z)}{q'(z)}$$

Example AE-611

$$\frac{z+5}{z^3-5z^2-2z+24}=\frac{z+5}{(z-4)(z+2)(z-3)}=\frac{A}{z-4}+\frac{B}{z+2}+\frac{C}{z-3}$$

$$A=\lim_{z\to 4}(z-4)\frac{p(z)}{q(z)}=\lim_{z\to 4}(z-4)\frac{z+5}{(z-4)(z-3)(z+2)}$$

$$=\lim_{z\to 4}\frac{z+5}{(z-3)(z+2)}=\frac{4+5}{(4-3)(4+2)}=\frac{9}{6}=1.5$$

$$B=\lim_{z\to -2}(z-(-2))\frac{p(z)}{q(z)}=\lim_{z\to -2}(z+2)\frac{z+5}{(z-4)(z-3)(z+2)}$$

$$=\lim_{z\to -2}\frac{z+5}{(z-3)(z-4)}=\frac{-2+5}{(-2-3)(-2-4)}=\frac{3}{30}=0.1$$

$$C=\lim_{z\to 3}(z-3)\frac{p(z)}{q(z)}=\lim_{z\to 3}\frac{z+5}{(z-4)(z+2)}=\frac{3+5}{(3-4)(3+2)}=-1.6$$

Using the Second residue Formula

In our problem $p(z)=z+5$ $q(z)=z^3-5z^2-2z+24$ and $q'(z)=3z^2-10z-2$

$$A=\lim_{z\to 4}\frac{z+5}{3z^2-10z-2}=\frac{4+5}{3\ 4^2-10\ 4-2}=\frac{9}{6}=1.5$$

$$B=\lim_{z\to -2}\frac{z+5}{3z^2-10z-2}=\frac{-2+5}{3\ (-2)^2-10\ (-2)-2}=\frac{3}{30}=0.$$

$$C=\lim_{z\to 3}\frac{z+5}{3z^2-10z-2}=\frac{3+5}{3\ (3)^2-10\ (3)-2}=\frac{8}{-5}=$$

Note we get the same result by **classical method**

$$\frac{z+5}{z^3-5z^2-2z+24}=\frac{A}{z-4}+\frac{B}{z+2}+\frac{C}{z-3}$$

$$=\frac{A(z+2)(z-3)+B(z-4)(z-3)+C(z-4)(z+2)}{(z-4)(z+2)(z-3)}$$

$$=\frac{A(z^2-z-6)+B(z^2-7z-12)+C(z^2-2z-8)}{(z-4)(z+2)(z-3)}$$

$$=\frac{z^2(A+B+C)+z(-7A-B-2C)+12A-6B-8C}{(z-4)(z+2)(z-3)}$$

$$z^2(A+B+C)+z(-7A-B-2C)+12A-6B-8C=z+5$$

$A+B+C=0,\quad -7A-B-2C=1,\quad 12A-6B-8C=$

Solving for A,B,C we get A=1.5 B=0.1 C=-1.6

$$\frac{z+5}{z^3-5z^2-2z+24}=\frac{1.5}{z-4}+\frac{0.1}{z+2}+\frac{-1.6}{z-3}$$

Example AE-612

$$\frac{2z+12}{z^2+2z+2}=\frac{2z+12}{[z-(-1+i)][z-(-1-i)]}=\frac{2z+12}{(z+1-i)(z+1+}$$

$$=\frac{A}{(z+1-i)}+\frac{B}{(z+1+i)}$$

Using the first residue formula

$$A=\lim_{z\to -1+i}(z+1-i)\frac{2z+12}{(z+1-i)(z+1+i)}=\lim_{z\to -1+i}\frac{2z+12}{(z+1+i)}$$

$$=\frac{2(-1+i)+12}{((-1+i)+1+i)}=\frac{2i+10}{2i}=\frac{2i}{2i}+\frac{10}{2i}=1-5i$$

$$B=\lim_{z\to -1-i}(z+1+i)\frac{2z+12}{(z+1-i)(z+1+i)}=\lim_{z\to -1-i}\frac{2z+12}{(z+1-i)}=1+5i$$

Using the second residue formula

$p(z)=2z+12,\quad q(z)=z^2+2z+2,\quad q'(z)=2z+2$

$$A=\lim_{z\to -1+i}\frac{p(z)}{q'(z)}=\frac{2z+12}{2z+2}=\frac{2(-1+i)+12}{2(-1+i)+2}=1-5i$$

$$B=\lim_{z\to -1-i}\frac{p(z)}{q'(z)}=\frac{2z+12}{2z+2}=\frac{2(-1-i)+12}{2(-1-i)+2}=1+5i$$

Using classical Method

$$\frac{2z+12}{z^2+2z+2}=\frac{A}{(z+1-i)}+\frac{B}{(z+1+i)}=\frac{A(z+1+i)+B(z+1-i)}{(z+1-i)(z+1+i)}$$

$(A+B)=2\quad A(1+i)+B(1-i)=12$ solution is $A=1-5i$, $B=1+5i$

Result $$\frac{2z+12}{z^2+2z+2}=\frac{1+5i}{(z+1-i)}+\frac{1-5i}{(z+1+i)}$$

EX.- 28 $$f(s) = \frac{s+3}{(s+1)(s+2)} = \frac{s+3}{s^2+3s+2} \quad f(t) = ?$$ 335

Solution

$$\frac{s+3}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$A = (s+1)\, f(s)\Big|_{s=-1} = (s+1)\, \frac{s+3}{(s+1)(s+2)}\Big|_{s=-1} = \frac{s+3}{s+2}\Big|_{s=-1} = \frac{-1+3}{-1+2} = 2$$

$$\boxed{A = 2}$$

$$B = (s+2)\, f(s)\Big|_{s=-2} = (s+2)\, \frac{s+3}{(s+1)(s+2)}\Big|_{s=-2} = \frac{s+3}{s+1}\Big|_{s=-2} = \frac{-2+3}{-2+1} = -1$$

$$B = -1$$

$$f(s) = \frac{s+3}{(s+1)(s+2)} = \frac{2}{s+1} + \frac{-1}{s+2}$$

$$f(t) = 2e^{-t} - e^{-2t}$$

$$\mathcal{L}^{-1}\left\{\frac{a+bi}{s-(x+iy)}+\frac{a-bi}{s-(x-iy)}\right\}=e^{xt}\left[2a\cos yt-2b\sin yt\right]$$

$$\mathcal{L}^{-1}\left\{\frac{10s+14}{s^2+6s+25}\right\}=\mathcal{L}^{-1}\left\{\frac{5+2i}{s-(-3+4i)}+\frac{5-2i}{s-(-3-4i}\right\}=e^{-3t}\left(2\times 5\cos 4t-2\times 2\sin 4t\right)$$

$$=e^{-3t}\left(10\cos 4t-4\sin 4t\right)$$

$$\mathcal{L}^{-1}\left\{\frac{6s+2}{s^2+6s+25}\right\}=\mathcal{L}^{-1}\left\{\frac{3+2i}{s-(-3+4i)}+\frac{3-2i}{s-(-3-4i)}\right\}=e^{-3t}\left(2\times 3\cos 4t-2\times 2\sin 4t\right)$$

$$=e^{-3t}\left(6\cos 4t-4\sin 4t\right)$$

$$\mathcal{L}^{-1}\left\{\frac{4s-22}{s^2+10s+34}\right\}=\mathcal{L}^{-1}\left\{\frac{2+7i}{s-(-5+3i)}+\frac{2-7i}{s-(-5-3i}\right\}=e^{-5t}\left(2\times 2\cos 3t-2\times 7\sin 3t\right)$$

$(+3i)$ $(+7i)$

$$=e^{-5t}\left(4\cos 3t-14\sin 3t\right)$$

$$\mathcal{L}^{-1}\left\{\frac{4s+62}{s^2+10s+34}\right\}=\mathcal{L}^{-1}\left\{\frac{2-7i}{s-(-5+3i)}+\frac{2+7i}{s-(-5-3i)}\right\}=e^{-5t}\left(2\times 2\cos 3t-2\times(-7)\sin 3t\right)$$

$(+3i)$ $(-7i)$

$$=e^{-5t}\left(4\cos 3t+14\sin 3t\right)$$

$$\mathcal{L}^{-1}\left\{\frac{4s+22}{s^2-10s+34}\right\}=\mathcal{L}^{-1}\left\{\frac{2-7i}{s-(5+3i)}+\frac{2+7i}{s-(5-3i)}\right\}=e^{5t}\left(2\times 2\cos 4t-2\times(-7)\sin 4t\right)$$

$$=e^{5t}\left(4\cos 4t+14\sin 4t\right)$$

$$\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathcal{L}\{e^{-at}\cos bt\} = \frac{s+a}{(s+a)^2+b^2}$$

$$\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2} \qquad \mathcal{L}\{e^{-at}\sin bt\} = \frac{b}{(s+a)^2+b^2}$$

EX-34

$$\frac{10s+14}{s^2+6s+25} \qquad s^2+6s+25=0 \quad \begin{array}{l} s_1=-3+4i \\ s_2=-3-4i \end{array}$$

$$s^2+6s+25=(s+3)^2+4^2$$

$$14 = 10 \times 3 + x$$
$$x = 14-30=-16$$

$$\frac{10s+14}{s^2+6s+25} = \frac{10s+14}{(s+3)^2+4^2} = 10\frac{(s+3)}{(s+3)^2+4^2} + \frac{-16}{(s+3)^2+4^2}$$

$$= 10\frac{s+3}{(s+3)^2+4^2} + (-4)\frac{4}{(s+3)^2+4^2}$$

$$\downarrow$$

$$10e^{-3t}\cos 4t + (-4)\,e^{-3t}\sin 4t$$

$$= e^{-3t}(10\cos 4t - 4\sin 4t)$$

EX-35

$$2 = 6 \times 3 + x$$
$$2-18 = x$$

$$\frac{6s+2}{s^2+6s+25} = \frac{6s+2}{(s+3)^2+4^2} = 6\frac{s+3}{(s+3)^2+4^2} + \frac{-16}{(s+3)^2+4^2} = 6\frac{s+3}{(s+3)^2+4^2} - 4\frac{4}{(s+3)^2+4^2}$$

$$\downarrow$$

$$6e^{-3t}\cos 4t - 4e^{-3t}\sin 4t$$

$$= e^{-3t}(6\cos 4t - 4\sin 4t)$$

EX-36

$$\mathcal{L}^{-1}\left\{\frac{4s-22}{s^2+10s+34}\right\}=?$$

$$s^2+10s+34=0 \rightarrow s_1=-5+3i,\quad s_2=-5-3i$$

$$-22=4\times5+x$$
$$-22-20=x$$

$$\frac{4s-22}{s^2+10s+34}=\frac{4s-22}{(s+5)^2+3^2}=4\frac{s+5}{(s+5)^2+3^2}+\frac{-42}{(s+5)^2+3^2}$$

$$=4\frac{s+5}{(s+5)^2+3^2}-14\frac{3}{(s+5)^2+3^2}$$

$$\swarrow$$

$$4e^{-5t}\cos3t-14e^{-5t}\sin3t$$

$$=e^{-5t}(4\cos3t-14\sin3t)$$

EX-37

$$\mathcal{L}^{-1}\left\{\frac{4s+22}{s^2-10s+34}\right\}$$

$$s^2-10s+34=0 \rightarrow s_1=5+3i,\quad s_2=5-3i$$

$$22=4\times(-5)+x$$
$$22+20=x$$

$$\frac{4s+22}{s^2-10s+34}=\frac{4s+22}{(s-5)^2+3^2}=4\frac{s-5}{(s-5)^2+3^2}+\frac{42}{(s-5)^2+3^2}$$

$$=4\frac{s-5}{(s-5)^2+3^2}+14\frac{3}{(s-5)^2+3^2}$$

$$\swarrow$$

$$4e^{5t}\cos3t+14e^{5t}\sin3t$$

$$=e^{5t}(4\cos3t+14\sin3t)$$

EX-41

s-345

$$F(s)=\frac{13s^2+82s+122}{s^3+9s^2+26s+24}=\frac{1}{s+4}+\frac{7}{s+3}+\frac{5}{s+2}$$

$$\downarrow$$

$$f(t)=e^{-4t}+7e^{-3t}+5e^{-2t}$$

EX-42

$$F(s)=\frac{17s^4-16s^3+497s^2-1200s+10978}{s^5-4s^4+23s^3-416s^2+2966s-4420}$$

$$=\frac{3}{s-(-4+7i)}+\frac{3}{s-(-4-7i)}+\frac{2-7i}{s-(5+3i)}+\frac{2+7i}{s-(5-3i)}+\frac{7}{s-2}$$

$$f(t)=e^{-4t}(2\times 3\cos 7t-2\times 0\sin 3t)+e^{5t}(2\times 2\cos 3t-2\times(-7)\sin 3t)+7e^{2t}$$

$$=6e^{-4t}\cos 7t+e^{5t}(4\cos 3t+14\sin 3t)+7e^{2t}$$

Repeated real roots

$$F(s) = \frac{10}{(s+2)} \longrightarrow f(t) = 10\,e^{-2t}$$

$$F(s) = \frac{10}{(s+2)^2} \longrightarrow f(t) = 10\,t\,e^{-2t}$$

$$F(s) = \frac{10}{(s+2)^3} \longrightarrow f(t) = 10\,t^2\,e^{-2t}$$

$$F(s) = \frac{2}{s+4} + \frac{3}{(s+4)^2} \longrightarrow f(t) = 2\,e^{-4t} + 3\,t\,e^{-4t}$$

$$F(s) = \frac{2}{s+2} + \frac{3}{(s+4)^2} + \frac{5}{(s+1)^3} \rightarrow f(t) = 2\,e^{-2t} + 3\,t\,e^{-4t} + \frac{5}{2}t^2 e^{-t}$$

$$F(s) = \frac{1}{s} + \frac{2}{s^2} + \frac{3}{s+1} \longrightarrow f(t) = u(t) + t\,u(t) + 3\,e^{-t}$$

$$F(s) = \frac{100(s+25)}{s(s+5)^3} = \frac{A}{s} + \frac{B}{(s+5)} + \frac{C}{(s+5)^2} + \frac{D}{(s+5)^2}$$

$A = 20$ $\quad B = -400$ $\quad C = -100$ $\quad D = -20$

$$F(s) = \frac{20}{s} - \frac{400}{s+5} - \frac{100}{(s+5)^2} - \frac{20}{(s+5)^3}$$

$$f(t) = 20\,u(t) - 400\,e^{-5t} - 100\,t\,e^{-5t} - \frac{20}{2}t^2 e^{-5t}$$

How can we calculate A, B, C, D

$$F(s) = \frac{ms+n}{(s+a)^2(s+b)} = \frac{A}{s+a} + \frac{B}{(s+a)^2} + \frac{C}{s+b}$$

$$C = (s+b)\,F(s)\Big|_{s=-b} = (s+b)\,\frac{ms+n}{(s+a)^2(s+b)}\Big|_{s=-b} = \frac{ms+n}{(s+a)^2}\Big|_{s=-b}$$

$$= \frac{m(-b)+n}{(-b+a)^2}$$

$$-\quad B = (s+a)^2\,F(s)\Big|_{s=-a} = (s+a)^2\,\frac{ms+n}{(s+a)^2(s+b)}\Big|_{s=-a} = \frac{ms+n}{s+b}\Big|_{s=-a} = \frac{-ma+n}{-a+b}$$

$$A = \frac{d}{ds}\left[(s+a)^2 F(s)\right]\Big|_{s=-a} = \frac{d}{ds}\left[\frac{ms+n}{s+b}\right] = \frac{(ms+n)'(s+b)-(ms+n)(s+b)'}{(s+b)^2}$$

$$= \frac{m(s+b)-(ms+n)}{(s+b)^2} =$$

$$F(s) = \frac{6s^2+17s+14}{(s+1)^2(s+2)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+2}$$

$$C = \frac{6s^2+17s+14}{(s+1)^2}\Big|_{s=-2} = \frac{6\times(-2)^2+17(-2)+14}{(-2+1)^2} = 4$$

$$B = \frac{6s^2+17s+14}{s+2}\Big|_{s=-1} = \frac{6(-1)^2+17(-1)+14}{-1+2} = 3$$

$$A = \frac{d}{ds}\left[\frac{6s^2+17s+14}{s+2}\right]\Big|_{s=-1} = \frac{(12s+17)(s+2)-1(6s^2+17s+14)}{(s+2)^2}\Big|_{s=-1} = 2$$

$$\frac{15z^3+1}{(z+2)^3(z-1)^2(z+10)}=\frac{A_1}{(z+2)}+\frac{A_2}{(z+2)^2}+\frac{A_3}{(z+2)^3}+\frac{B_1}{(z-1)}+\frac{B_2}{(z-1)^2}+\frac{C}{(z+10)}$$

$$C=(z+10)F(z)\Big|_{z=-10}=\frac{15z^3+1}{(z+2)^3(z-1)^2}\Big|_{z=-10}$$

$$=\frac{15(-10)^3+1}{(-10+2)^3(-10-1)^2}=0.2421$$

$$B_2=(z-1)^2F(z)\Big|_{z=1}=\frac{15z^3+1}{(z+2)^3(z+10)}\Big|_{z=1}$$

$$=\frac{15(1)^3+1}{(1+2)^3(1+10)}=0.0539$$

$$B_1=\frac{d}{dz}\left[(z-1)^2F(z)\right]\Big|_{z=1}=\frac{d}{dz}\left[\frac{15z^3+1}{(z+2)^3(z+10)}\right]_{z=1}$$

$p=15z^3+1 \qquad p'=45z^2$

$q=(z+2)^3(z+10) \qquad q'=3(z+2)^2(z+10)+(z+2)^3\,1$

$q'=4z^3+48z^2+144z+128$

$$\left(\frac{p}{q}\right)'=\left(\frac{p'q-pq'}{q^2}\right)$$

$$=\frac{-15z^6+1080z^4+3836z^3+3552z^2-144z-128}{(z+2)^6(z+10)^2}$$

$$\frac{-15z^6+1080z^4+3836z^3+3552z^2-144z-128}{(z+2)^6(z+10)^2}\Big|_{z=1}$$

$$\frac{-15\,1^6+1080\,1^4+3836\,1^3+3552\,1^2-144\,1-128}{(1+2)^6(1+10)^2}=0.092$$

$B_1=0.092$

$$A_3=(z+2)^3F(z)\Big|_{z=-2}=\frac{15z^3+1}{(z-1)^2(z+10)}\Big|_{z=-2}$$

$$=\frac{15(-2)^3+1}{(-2-1)^2(-2+10)}=-1.6528$$

$$A_2=\frac{d}{dz}\left[(z+2)^3F(z)\right]\Big|_{z=-2}=\frac{d}{dz}\left[\frac{15z^3+1}{(z-1)^2(z+10)}\right]_{z=-2}$$

$p=15z^3+1 \qquad p'=45z^2$

$q=(z-1)^2(z+10) \qquad q'=2(z-1)(z+10)+(z-1)^2\,1$

$q'=3z^2+16z-19$

$A_2=1.604$

$$A_1=\frac{d^2}{dz^2}\left[\frac{15z^3+1}{(z-1)^2(z+10)}\right]\Big|_{z=-2}=-0.33$$

$$\frac{15z^3+1}{(z+2)^3(z-1)^2(z+10)}=$$

$$\frac{-0.33}{z+2}+\frac{1.604}{(z+2)^2}+\frac{-1.652}{(z+2)^3}+\frac{0.092}{z-1}$$

$$+\frac{0.0539}{(z-1)^2}+\frac{0.2421}{z+10}$$

Improper Rational functions

$f(s) = \frac{s^2+7s+5}{s+3}$ → improper

numerator → s^2 (2)
denominator → s (1)
$2 > 1$

$$\begin{array}{r|l} & s+4 \\ \hline s+3 & s^2+7s+5 \\ & -(s^2+3s) \\ \hline & 0 \quad 4s+5 \\ & -(4s+12) \\ \hline & -7 \end{array}$$

$$\frac{s^2+7s+5}{s+3} = s+4+\frac{-7}{s+3}$$

$$\begin{array}{r|l} & 3 \\ \hline 5 & 17 \\ & -15 \\ \hline & 2 \end{array}$$

$$\frac{17}{5} = 3 + \frac{2}{5}$$

$f(t) = \delta(t) \longrightarrow F(s) = 1$

$f(t) = \frac{d}{dt}\delta(t) \longrightarrow F(s) = s$

$f(t) = \frac{d^2}{dt^2}\delta(t) \longrightarrow F(s) = s^2$

$$F(s) = \frac{s^2+7s+5}{s+3} = s+4+\frac{-7}{s+3}$$

$$f(t) = \frac{d}{dt}\delta(t) + 4\delta(t) - 7e^{-3t}$$

$$F(s) = \frac{s^3 + 7s^2 + 9s + 7}{s^2 + 3s + 2}$$

$$\begin{array}{r|l} & s + 4 \\ \hline s^2 + 3s + 2 & s^3 + 7s^2 + 9s + 7 \\ & s^3 + 3s^2 + 2s \\ - & \overline{ 4s^2 + 7s + 7} \\ & 4s^2 + 12s + 8 \\ - & \overline{ -5s - 1} \end{array}$$

$$F(s) = \frac{s^3 + 5s^2 + 9s + 7}{s^2 + 3s + 2} = s + 4 + \frac{-5s - 1}{s^2 + 3s + 2}$$

$$\frac{-5s - 1}{s^2 + 3s + 2} = \frac{-5s - 1}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1}$$

$$A = \left.\frac{-5s - 1}{s+1}\right|_{s=-2} = \frac{-5(-2) - 1}{-2+1} = -9$$

$$B = \left.\frac{-5s - 1}{s+2}\right|_{s=-1} = \frac{-5(-1) - 1}{-1+2} = 4$$

$$F(s) = s + 4 + \frac{-9}{s+2} + \frac{4}{s+1}$$

$$f(t) = \frac{d}{dt}\delta(t) + 4\delta(t) - 9e^{-2t} + 4e^{-t}$$

352.

$$F(s) = \frac{s^4 + 13s^3 + 66s^2 + 200s + 300}{s^2 + 9s + 20}$$

$$
\begin{array}{r|l}
 & s^2 + 4s + 10 \\
\hline
s^2 + 9s + 20 & s^4 + 13s^3 + 66s^2 + 200s + 300 \\
 & -(s^4 + 9s^3 + 20s^2) \\
\hline
 & 4s^3 + 46s^2 + 200s + 300 \\
 & -(4s^3 + 36s^2 + 80s) \\
\hline
 & 10s^2 + 120s + 300 \\
 & -(10s^2 + 90s + 200) \\
\hline
 & 30s + 100
\end{array}
$$

$$F(s) = s^2 + 4s + 10 + \frac{30s + 100}{s^2 + 9s + 20}$$

$$\frac{30s + 100}{s^2 + 9s + 20} = \frac{30s + 100}{(s+4)(s+5)} = \frac{-20}{s+4} + \frac{50}{s+5}$$

$$F(s) = s^2 + 4s + 10 + \frac{-20}{s+4} + \frac{50}{s+5}$$

$$f(t) = \frac{d^2}{dt^2}\delta(t) + 4\frac{d}{dt}\delta(t) + 10\,\delta(t) - 20e^{-4t} + 50e^{-5t}$$