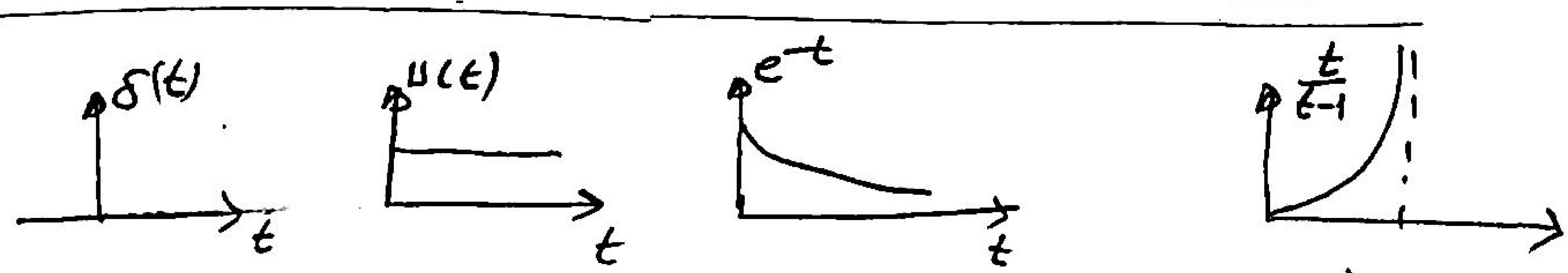
The Stability of Linear Feedback Systems

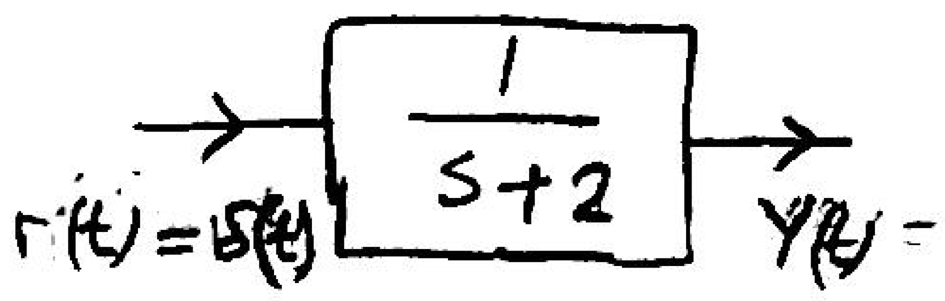
A stable system is defined as a system with a bounded (limited) system response. That is, if the system is subjected to a bounded input or disturbance and the response is bounded in magnitude, the system is said to be stable.

A stable system is a dynamic system with a bounded response to a bounded input.



Bounded inputs

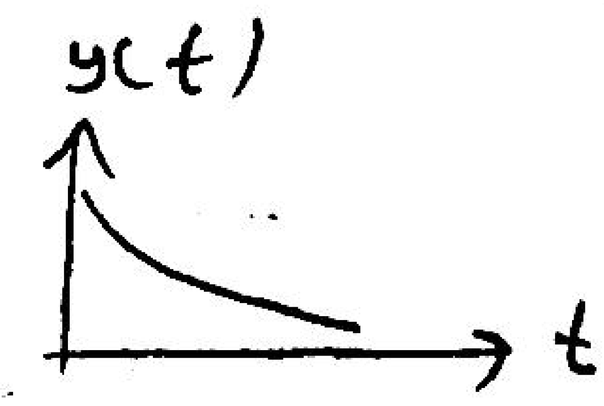
unbounded input



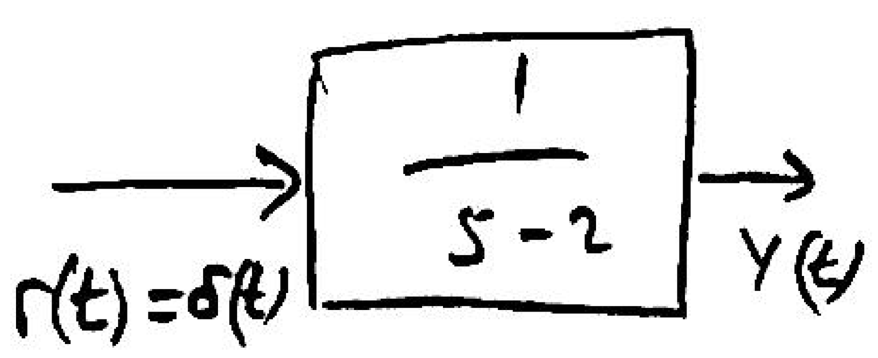
$R(s)=1$

$Y(s)=\frac{1}{s+2}R(s)=\frac{1}{s+2}$

$y(t)=e^{-2t}$

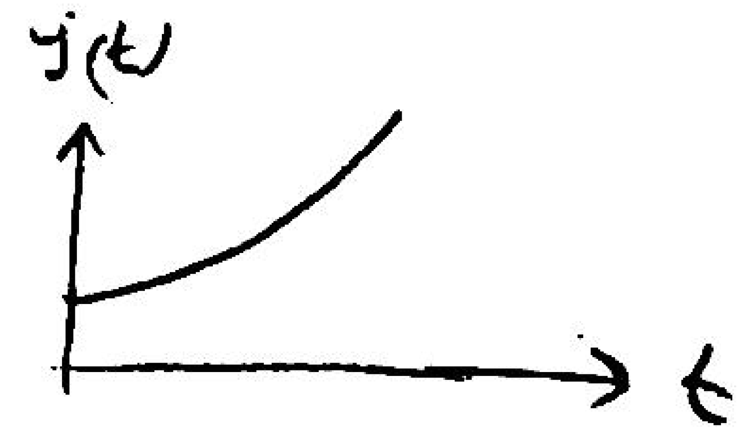


stable

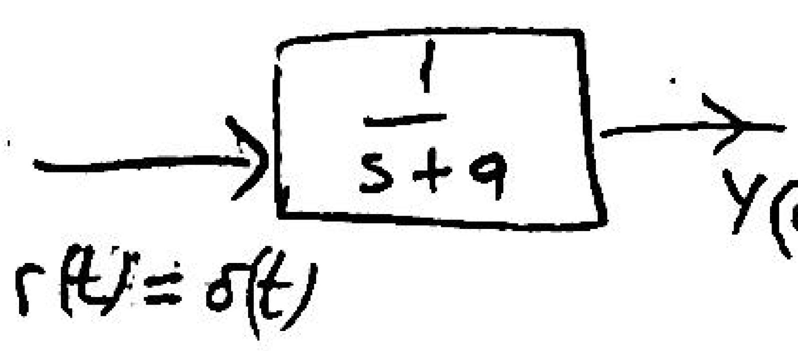


$Y(s)=\frac{1}{s-2}$

$y(t)=e^{2t}$



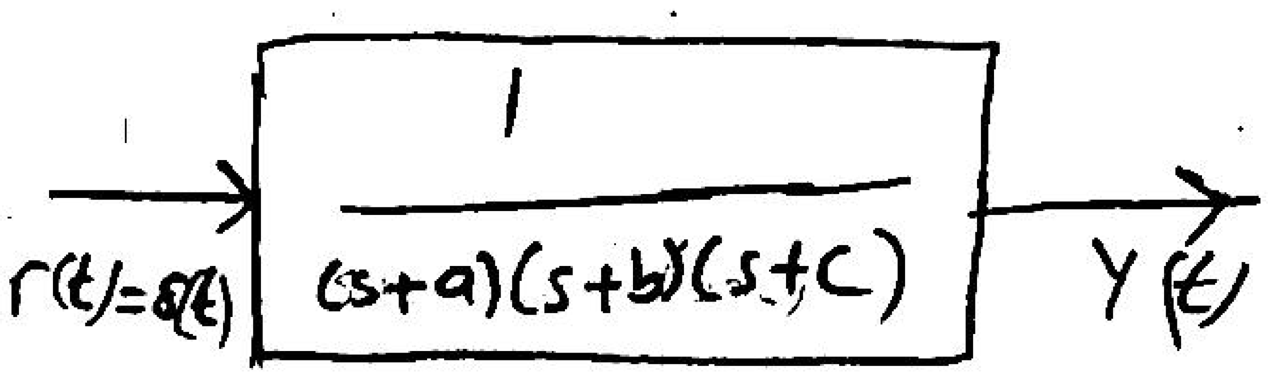
Unstable



$Y(s)=\frac{1}{s+a}$ $\quad y(t)=e^{-at}$

$a>0$ stable

$a<0$ Unstable



$$Y(s)=\frac{A}{s+a}+\frac{B}{s+b}+\frac{C}{s+c}$$

$$y(t)=Ae^{-at}+Be^{-bt}+Ce^{-ct}$$

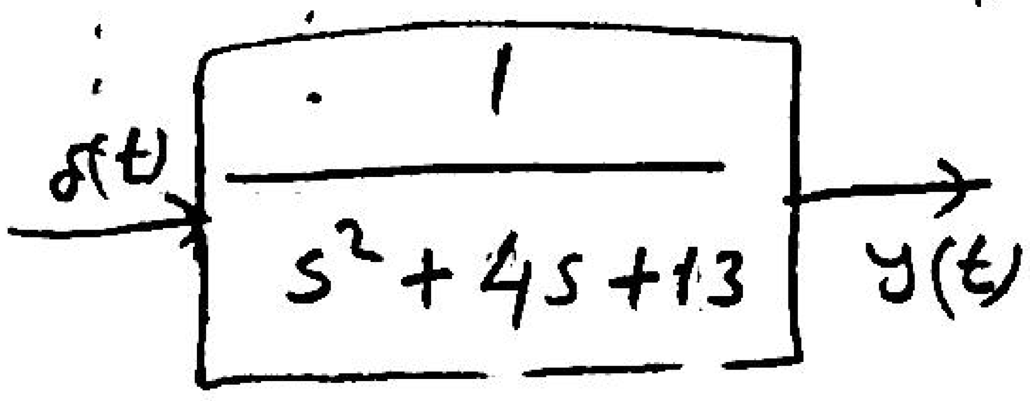
if $a>0$ $\quad b>0$ $\quad c>0$ stable

$\frac{1}{(s+1)(s+2)(s+3)(s+4)}$ stable

$\frac{1}{(s+1)(s+2)(s+3)(s-5)}$ Unstable

$\frac{1}{(s-1)(s-2)(s-3)}$ Unstable

$\frac{1}{(s+1)(s-1)(s+2)(s-2)}$ Unstable

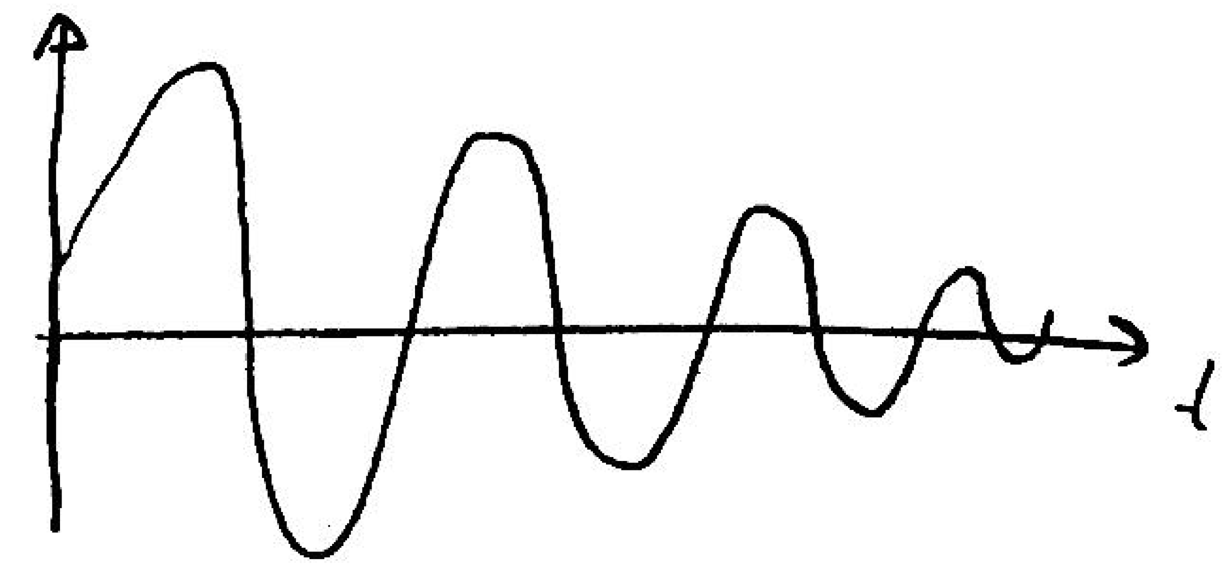


$$Y(s) = \frac{1}{s^2+4s+13}$$

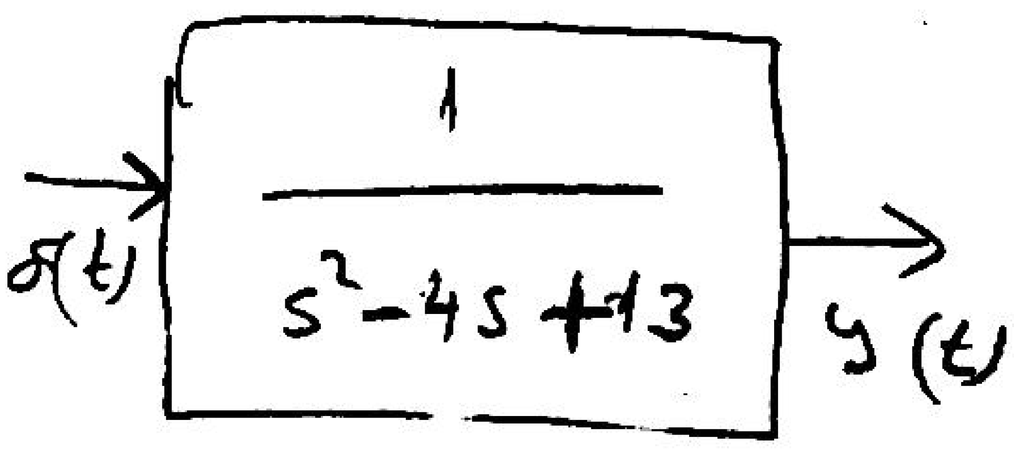
$s_1 = -2-3j$

$s_2 = -2+3j$

$$y(t) = e^{-2t}(A\cos 3t + B\sin 3t)$$



stable $y(\infty) = 0$

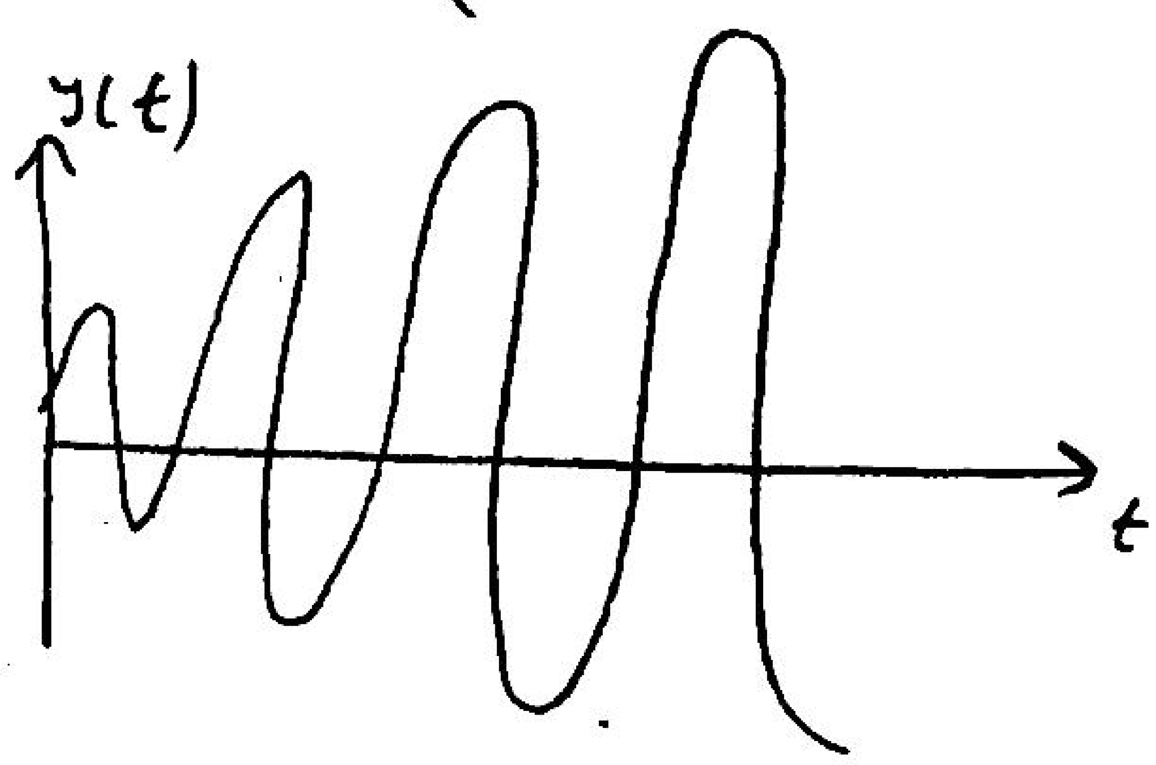


$$Y(s) = \frac{1}{s^2-4s+13}$$

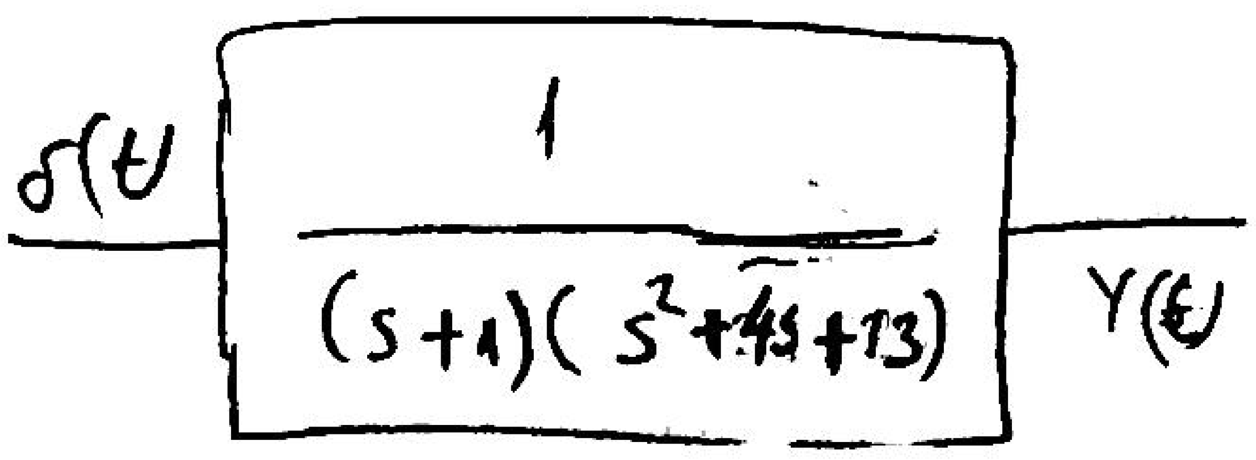
$s_1 = 2+3j$

$s_2 = 2-3j$

$$y(t) = e^{2t}(A\cos 3t + B\sin 3t)$$



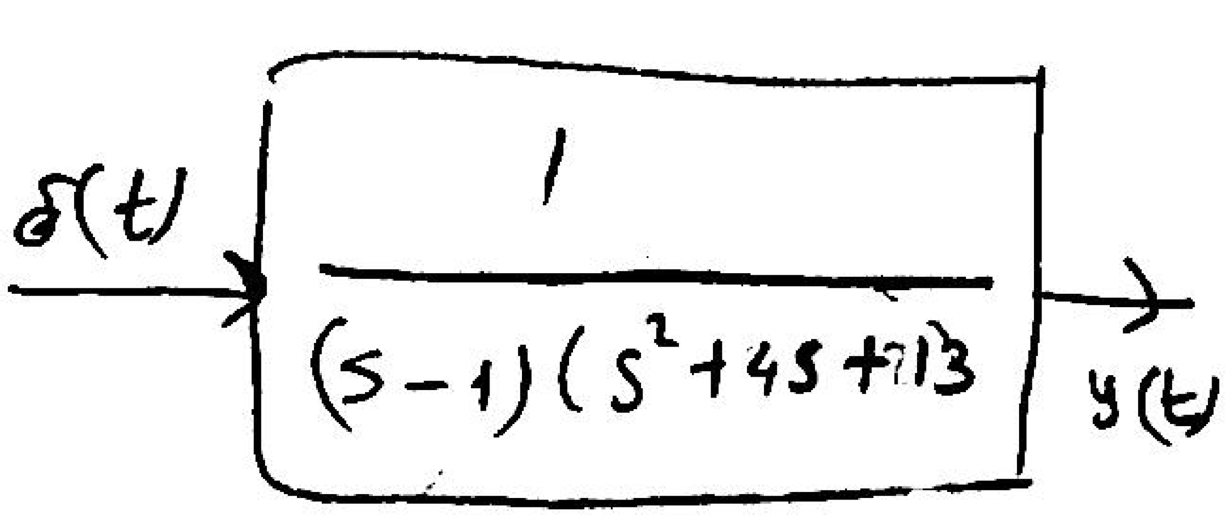
Unstable $y(\infty) = \infty$



$$Y(s) = \frac{1}{(s+1)(s^2+4s+13)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4s+13}$$

$$y(t) = Ae^{-t} + e^{-2t}(D\cos 3t + E\sin 3t)$$

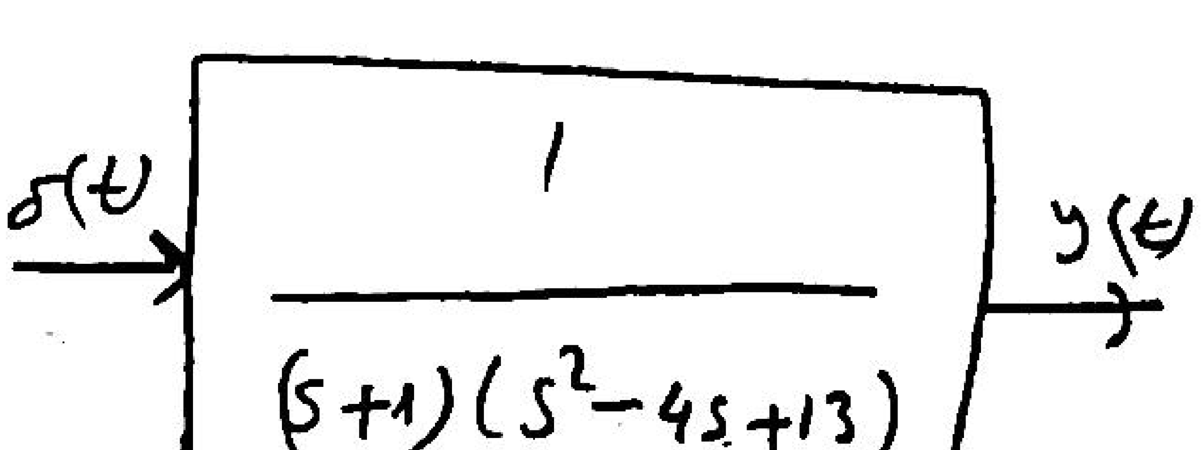
stable $y(\infty) = 0$



$$Y(s) = \frac{1}{(s-1)(s^2+4s+13)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+4s+13}$$

$$y(t) = Ae^{t} + e^{-2t}(D\cos 3t + E\sin 3t)$$

Unstable $y(\infty) = \infty$



$$Y(s) = \frac{1}{(s-1)(s^2-4s+13)} = \frac{A}{s-1} + \frac{Bs+C}{s^2-4s+13}$$

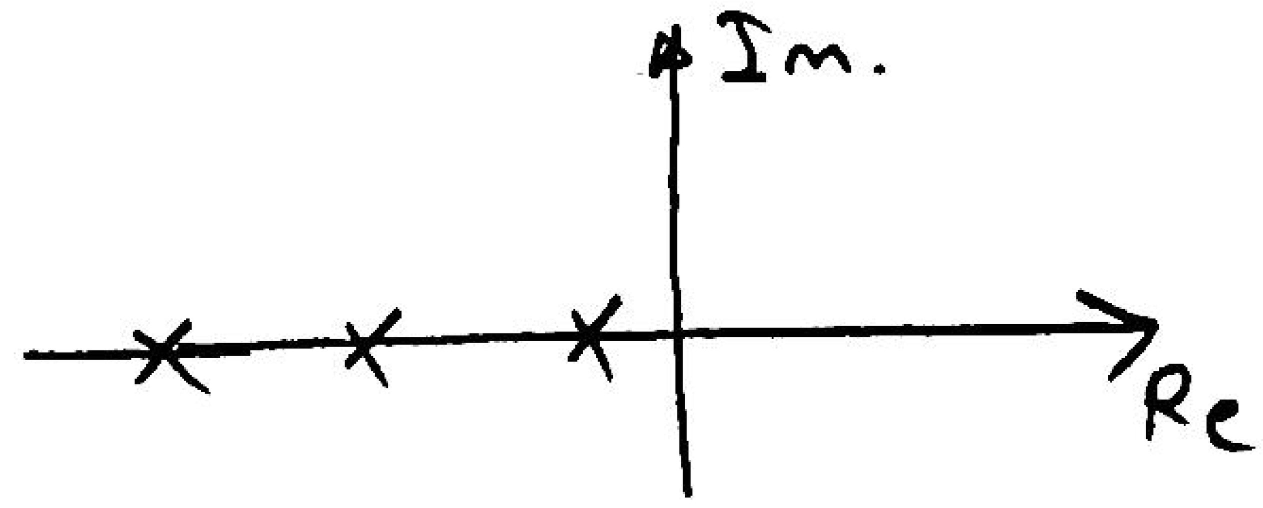
$$y(t) = Ae^{-t} + e^{2t}(D\cos 3t + E\sin 3t)$$

Unstable $y(\infty) = \infty$

. Thus a **necessary and sufficient condition for a feedback system to be stable is that all the poles of the system transfer function have negative real parts.** A system is stable if all the poles of the transfer function are in the left-hand *s*-plane. We will call a system not stable if not all the roots are in the left-hand plane.

$$\frac{1}{(s+1)(s+2)(s+3)}$$

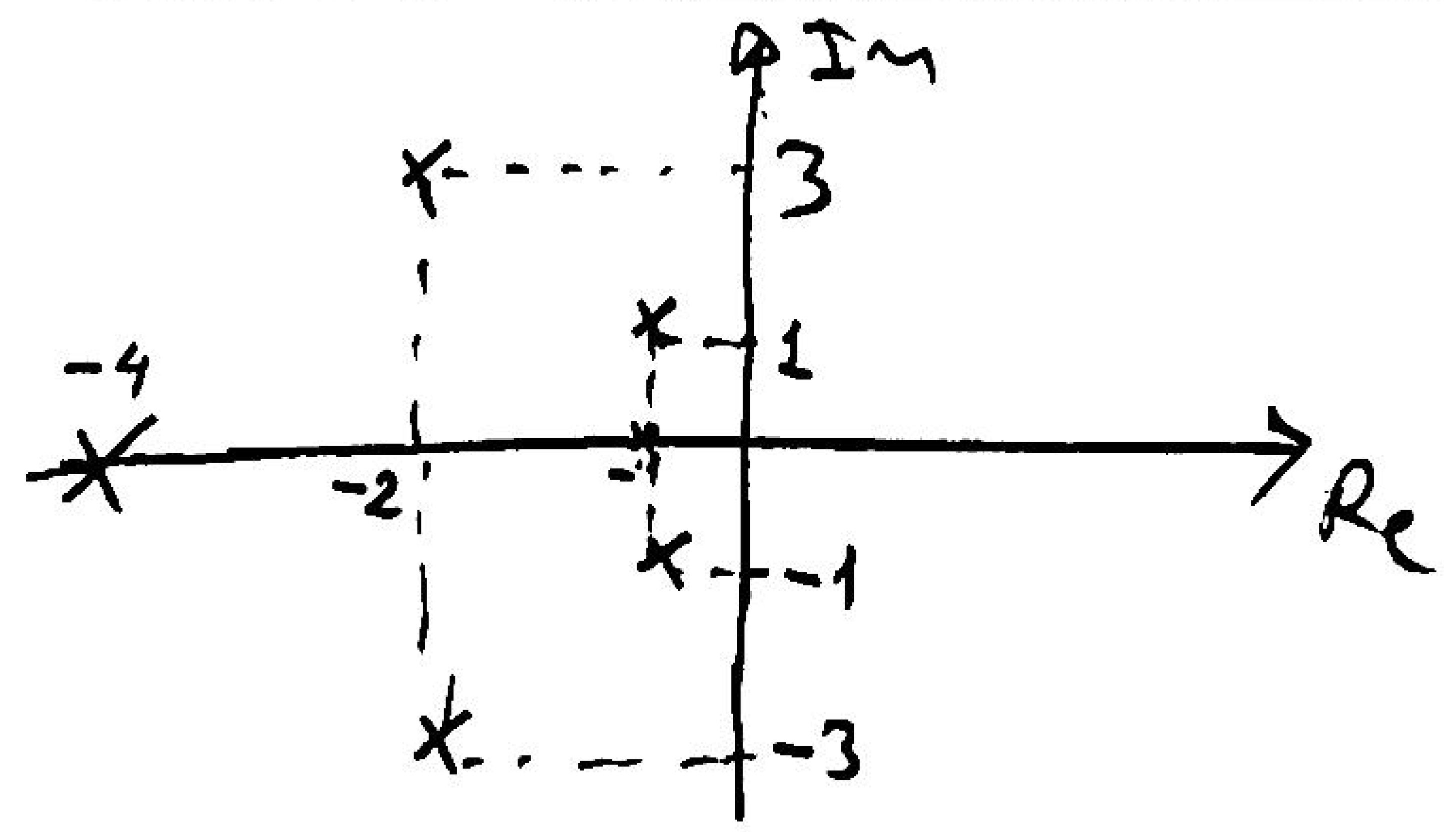
$s_1 = -1$
$s_2 = -2$
$s_3 = -3$



Stable.

$$\frac{1}{(s^2+2s+2)(s^2+4s+13)(s+4)}$$

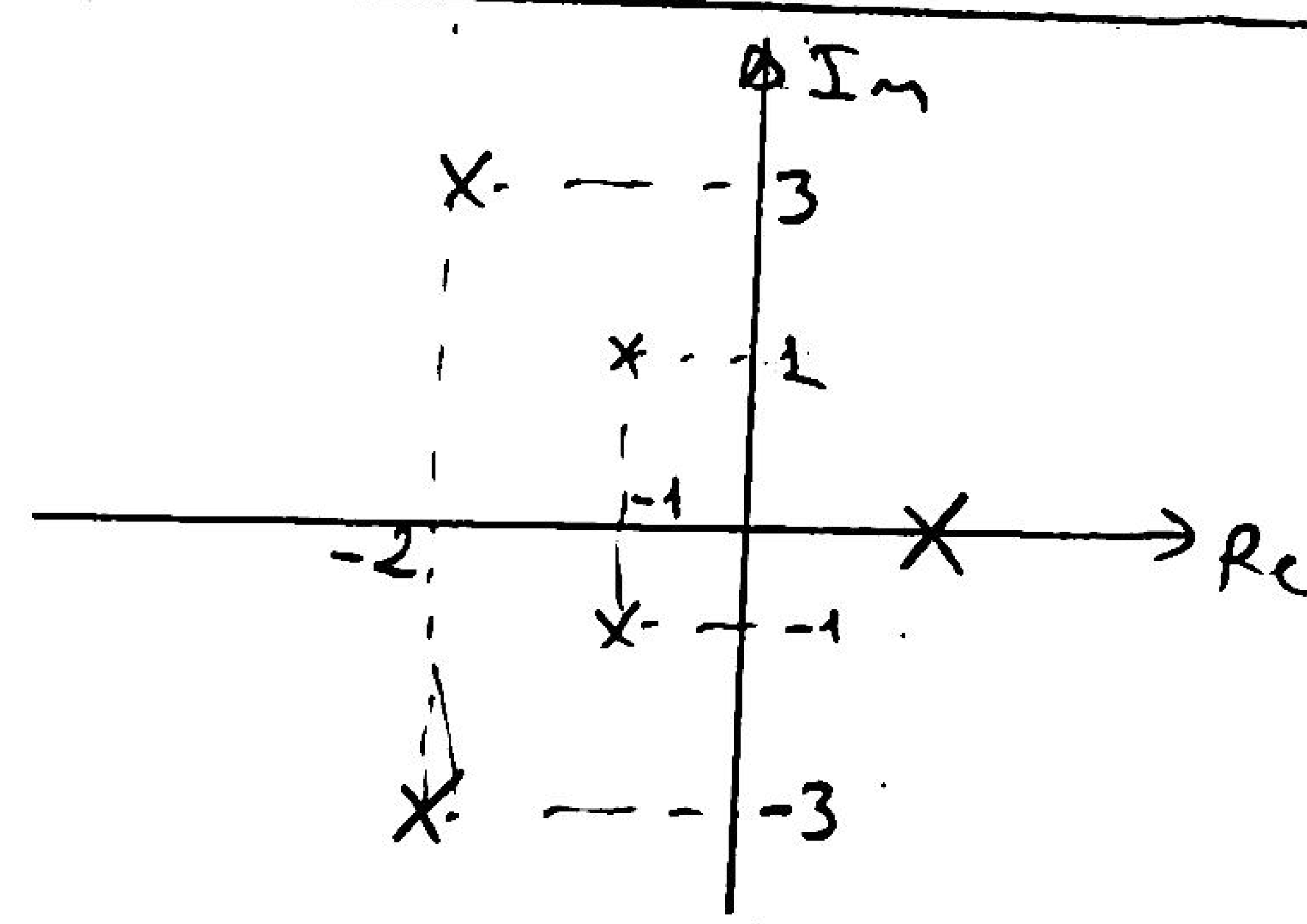
$s_1 = -1 - j$
$s_2 = -1 + j$
$s_3 = -2 - 3j$
$s_4 = -2 + 3j$
$s_5 = -2$



All the roots are in left half plane: Stable

$$\frac{1}{(s^2+2s+2)(s^2+4s+13)(s-1)}$$

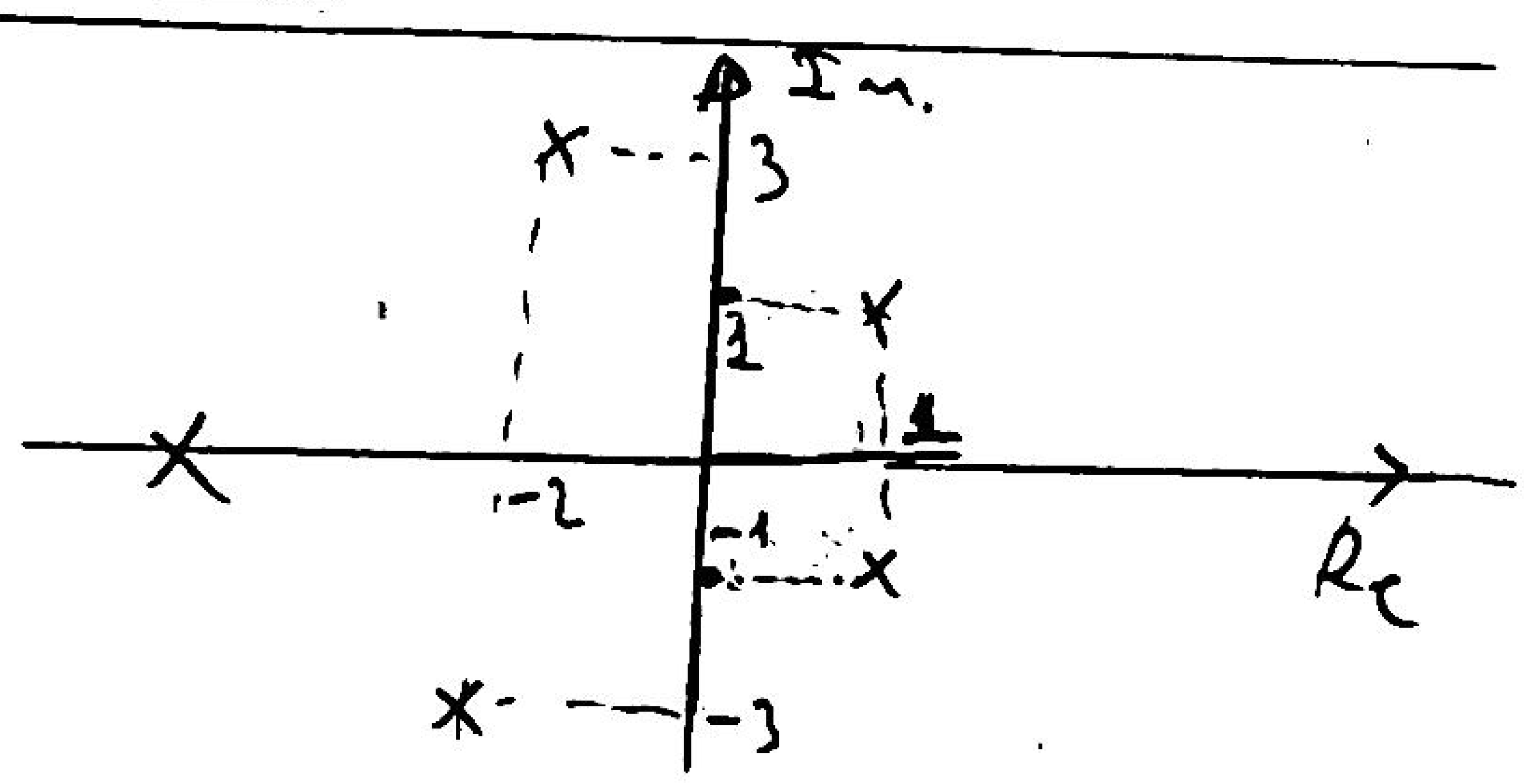
$s_1 = -1 - j$
$s_2 = -1 + j$
$s_3 = -2 - 3j$
$s_4 = -2 + 3j$
$s_5 = 1$



One root is on the right half plane Unstable

$$\frac{1}{(s^2-2s+2)(s^2+4s+13)(s+4)}$$

$s_1 = 1 - j$
$s_2 = 1 + j$
$s_3 = -2 - 3j$
$s_4 = -2 + 3j$
$s_5 = -4$



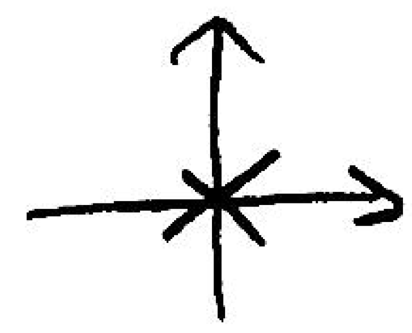
two roots are on the left

Unstable.

conditional stability

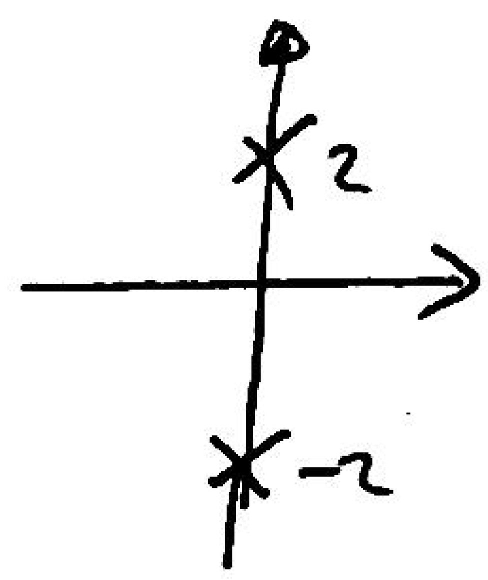
if there is a pole on the imaginary axis the system is conditionally stable.

$\frac{1}{s} \rightarrow s=0$



conditionaly stable

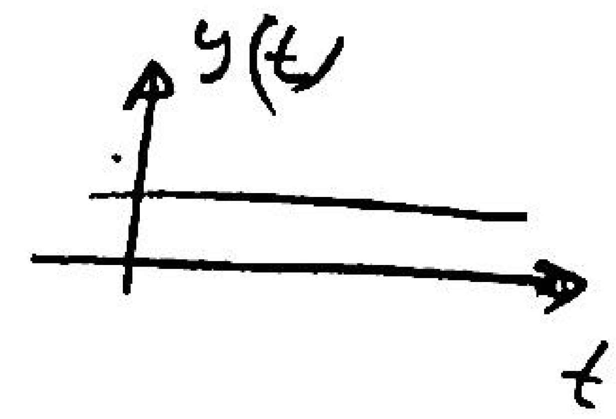
$\frac{1}{s^2+4} \rightarrow \begin{matrix} s_1=-2j \\ s_2=+2j \end{matrix}$



conditionally stable

$\delta(t) \rightarrow \boxed{\frac{1}{s}} \rightarrow y(t)$

$Y(s) = \frac{1}{s}$ $y(t) = u(t)$

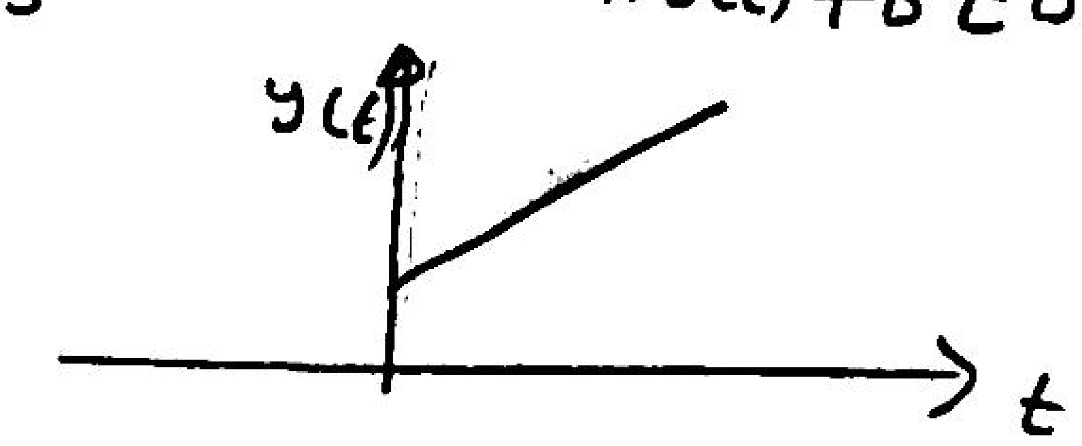


$y(\infty) \neq \infty$ stable

$u(t) \rightarrow \boxed{\frac{1}{s}} \rightarrow y(t)$

$Y(s) = \frac{1}{s}\cdot\frac{1}{s} = \frac{1}{s^2} = \frac{A}{s} + \frac{B}{s^2}$

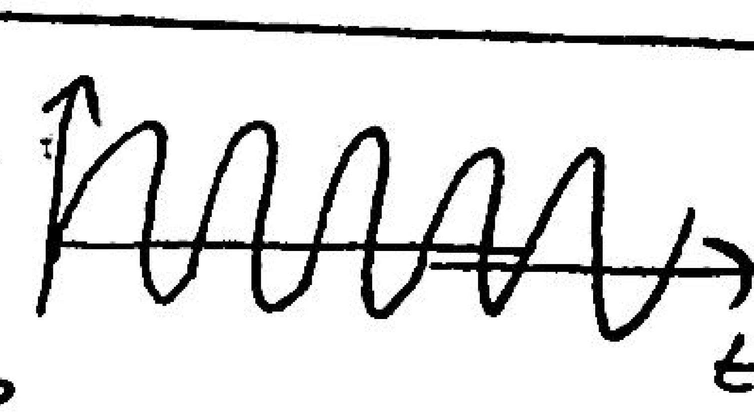
$y(t) = Au(t) + B\,t\,u(t)$



$y(\infty) = \infty$ <u>Unstable</u>

$\delta(t) \rightarrow \boxed{\frac{1}{s^2+4}} \rightarrow y(t)$

$Y(s) = \frac{1}{s^2+4}$ $y(t) = A\cos 2t + B\sin 2t$

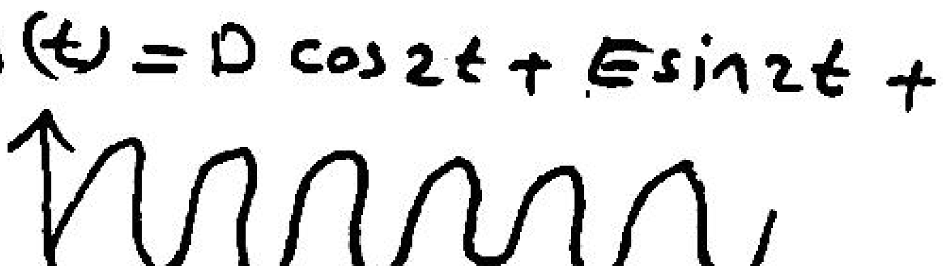


$y(\infty) \neq \infty$ stable

$u(t) \rightarrow \boxed{\frac{1}{s^2+4}} \rightarrow y(t)$

$Y(s) = \frac{1}{s^2+4}\,\frac{1}{s} = \frac{As+B}{s^2+4} + \frac{C}{s}$

$y(t) = D\cos 2t + E\sin 2t + Cu(t)$

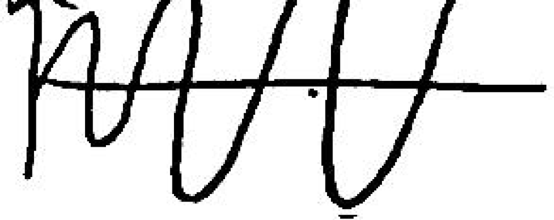


$y(\infty) \neq \infty$ <u>stable</u>

$\cos 2t \rightarrow \boxed{\frac{1}{s^2+4}} \rightarrow y(t)$

$Y(s) = \frac{1}{s^2+4}\,\frac{1}{s^2+4}$

$y(t) = A\cos 2t + B\sin 2t + t(C\cos 2t + D\sin 2t)$



$y(\infty) = \infty$ <u>Unstable</u>

(5)

$s^4 + 3s^3 + 15s^2 + 20s + 30 = 0$

$s^5 + 2s^4 + 3s^3 + 2s^2 + s + 5 = 0$

Is there any root on the right half plane (without finding the roots)

$$a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0 = 0$$

Further rows of the schedule are then completed as follows:

$$\begin{array}{c|ccc}
s^n & a_n & a_{n-2} & a_{n-4} \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} \\
s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} \\
s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} \\
\vdots & \vdots & \vdots & \vdots \\
s^0 & h_{n-1} & &
\end{array}$$

where

$$b_{n-1} = \frac{(a_{n-1})(a_{n-2}) - a_n(a_{n-3})}{a_{n-1}} = \frac{-1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix},$$

$$b_{n-3} = -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix},$$

and

$$c_{n-1} = \frac{-1}{b_{n-1}}\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix},$$

and so on. The algorithm for calculating the entries in the array can be followed on a determinant basis or by using the form of the equation for b_{n-1}.

The Routh–Hurwitz criterion states that the number of roots of q(s) with positive real parts is equal to the number of changes in sign of the first column of the Routh array. This criterion requires that there be no changes in sign in the first column for a stable system. This requirement is both necessary and sufficient.

Example.

$s^5 + 6s^4 + 14s^3 + 16s^2 + 9s + 2 = 0$

1	14	9	0
6	16	2	0
b_1	b_2	0	
c_1	c_2	0	
d_1	0		
e_1	0		

$b_1 = \frac{-1}{6}\begin{vmatrix} 1 & 14 \\ 6 & 16 \end{vmatrix} = -\frac{1}{6}(1\times16 - 6\times14) = 11.3$

$b_2 = \frac{-1}{6}\begin{vmatrix} 1 & 9 \\ 6 & 2 \end{vmatrix} = 8.66$

$c_1 = \frac{-1}{-11.3}\begin{vmatrix} 6 & 16 \\ 11.3 & 8.66 \end{vmatrix} = 11.44$

$c_2 = \frac{-1}{11.3}\begin{vmatrix} 6 & 2 \\ 11.3 & 0 \end{vmatrix} = 2$

$d_1 = \frac{-1}{11.44}\begin{vmatrix} 11.33 & 8.66 \\ 11.44 & 2 \end{vmatrix} = 6.68$

$e_1 = \frac{-1}{11.44}\begin{vmatrix} 11.4 & 2 \\ 6.68 & 0 \end{vmatrix} = 2$

Stable

The characteristic polynomial of a second-order system is

$$q(s) = a_2 s^2 + a_1 s + a_0.$$

The Routh array is written as

$$\begin{array}{c|cc} s^2 & a_2 & a_0 \\ s & a_1 & 0 \\ s^0 & b_1 & 0 \end{array}$$

where

$$b_1 = \frac{a_1 a_0 - (0)a_2}{a_1} = \frac{-1}{a_1}\begin{vmatrix} a_2 & a_0 \\ a_1 & 0 \end{vmatrix} = a_0.$$

Therefore the requirement for a stable second-order system is simply that all the coefficients be positive or all the coefficients be negative. ■

Third-order system

The characteristic polynomial of a third-order system is

$$q(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0.$$

The Routh array is

$$\begin{array}{c|cc} s^3 & a_3 & a_1 \\ s^2 & a_2 & a_0 \\ s^1 & b_1 & 0 \\ s^0 & c_1 & 0, \end{array}$$

where

$$b_1 = \frac{a_2 a_1 - a_0 a_3}{a_2}, \quad \text{and} \quad c_1 = \frac{b_1 a_0}{b_1} = a_0.$$

For the third-order system to be stable, it is necessary and sufficient that the coefficients be positive and $a_2 a_1 > a_0 a_3$. The condition when $a_2 a_1 = a_0 a_3$ results in a marginal stability case, and one pair of roots lies on the imaginary axis in the s-plane. This marginal case is recognized as Case 3 because there is a zero in the first column when $a_2 a_1 = a_0 a_3$, and it will be discussed under Case 3.

As a final example of characteristic equations that result in no zero elements in the first row, let us consider a polynomial

$$q(s) = (s - 1 + j\sqrt{7})(s - 1 - j\sqrt{7})(s + 3) = s^3 + s^2 + 2s + 24. \quad (6.9)$$

The polynomial satisfies all the necessary conditions because all the coefficients exist and are positive. Therefore utilizing the Routh array, we have

$$\begin{array}{c|rr} s^3 & 1 & 2 \\ s^2 & 1 & 24 \\ s^1 & -22 & 0 \\ s^0 & 24 & 0 \end{array}$$

Because two changes in sign appear in the first column, we find that two roots of $q(s)$ lie in the right-hand pl[illegible]

$$q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10. \tag{6.10}$$

The Routh array is then

$$\begin{array}{c|ccc}
s^5 & 1 & 2 & 11 \\
s^4 & 2 & 4 & 10 \\
s^3 & \epsilon & 6 & 0 \\
s^2 & c_1 & 10 & 0 \\
s^1 & d_1 & 0 & 0 \\
s^0 & 10 & 0 & 0
\end{array}$$

where

$$c_1 = \frac{4\epsilon - 12}{\epsilon} = \frac{-12}{\epsilon}, \quad \text{and} \quad d_1 = \frac{6c_1 - 10\epsilon}{c_1} \to 6.$$

There are two sign changes due to the large negative number in the first column, $c_1 = -12/\epsilon$. Therefore the system is unstable, and two roots lie in the right half of the plane.

Unstable system

As a final example of the type of Case 2, consider the characteristic polynomial

$$q(s) = s^4 + s^3 + s^2 + s + K, \tag{6.11}$$

where it is desired to determine the gain K that results in marginal stability. The Routh array is then

$$\begin{array}{c|ccc}
s^4 & 1 & 1 & K \\
s^3 & 1 & 1 & 0 \\
s^2 & \epsilon & K & 0 \\
s^1 & c_1 & 0 & 0 \\
s^0 & K & 0 & 0
\end{array}$$

where

$$c_1 = \frac{\epsilon - K}{\epsilon} \to \frac{-K}{\epsilon}.$$

Therefore for any value of K greater than zero, the system is unstable. Also, because the last term in the first column is equal to K, a negative value of K will result in an unstable system. Therefore the system is unstable for all values of gain K. ■

$$q(s) = s^3 + 2s^2 + 4s + K, \tag{6.12}$$

where K is an adjustable loop gain. The Routh array is then

$$\begin{array}{c|cc}
s^3 & 1 & 4 \\
s^2 & 2 & K \\
s^1 & \dfrac{8 - K}{2} & 0 \\
s^0 & K & 0
\end{array}$$

For a stable system, we require that

$$0 < K < 8.$$