Değişken dönüştürme

$\sqrt{a^2-x^2}$ şeklinde terim bulunduran ifadelerde $x = a \sin t \rightarrow dx = a \cos t$ dönüşümü yapılırsa karekökten kurtulur.

$$\sqrt{a^2-x^2} = \sqrt{a^2 - a^2 \sin^2 t} = a\sqrt{1-\sin^2 t}$$

$= a.\cos t$ olur.

fakat bir integrasyonda birden fazla değişik $\sqrt{b^2-x^2}$ varsa bu dönüşüm fayda vermez

$$\int \frac{x^2+3x+8}{\sqrt{16-x^2}}\,dx = ? \qquad \begin{aligned} x &= 4.\sin t \\ dx &= 4 \cos t\,dt \end{aligned}$$

$$= \int \frac{(4\sin t)^2 + 3(\sin t) + 8}{\sqrt{16-(4\sin t)^2}}\, 4\cos t\, dt$$

$$= \int \frac{16\sin^2 t + 3\sin t + 8}{4\cos t}\, 4\cos t\, dt = \int (16\sin^2 t + 3\sin t + 8)\, dt$$

çözülebilir

$$\int \frac{x^2+14+\sqrt{16-x^2}}{\sqrt{16-x^2}}\,dx \qquad \begin{aligned} x &= 4\sin t \\ dx &= 4\cos t\,dt \end{aligned}$$

$$= \int \frac{(4\sin t)^2+14+\sqrt{16-(4\sin t)^2}}{\sqrt{16-(4\sin t)^2}}\,4\cos t\,dt$$

$$= \int \frac{16\sin^2 t+14+4\cos t}{4\cos t}\,4\cos t\,dt$$

$$= \int \left(16\sin^2 t+14+4\cos t\right)dt$$

çözülebilir

$$\int \frac{x\sqrt{9-x^2}}{\sqrt{16-x^2}}\,dx \qquad \begin{aligned} x &= 4\sin t \\ dx &= 4\cos t\,dt \end{aligned}$$

$$\int \frac{4\sin t\sqrt{9-16\sin^2 t}}{4\cos t}\,4\cos t\,dt$$

$$\int \left(4\sin t\sqrt{9-16\sin^2 t}\right)dt \longrightarrow$$

kolay çözülebilen değil

bir faydası olmadı

$\sqrt{x^2-a^2}$ varsa

$$x=\frac{a}{\cos t} \rightarrow dx=\frac{a\sin t}{\cos^2 t}\,dt$$

$$\int \frac{x^2+8x+5}{\sqrt{x^2-16}}\,dx \qquad x=\frac{4}{\cos t} \qquad dx=\frac{4\sin t}{\cos^2 t}$$

$$\int \frac{(4\sin t)^2+8(4\sin t)+5}{\sqrt{\frac{4^2}{\cos^2 t}-16}}\; 4\,\frac{\sin t}{\cos^2 t}\,dt$$

$$\sqrt{\frac{16}{\cos^2 t}-16}=\sqrt{16\left(\frac{1-\cos^2 t}{\cos^2 t}\right)}=4\,\frac{\sin t}{\cos t}$$

$$=\int \frac{16\sin^2 t+32\sin t+5}{4\cdot\frac{\sin t}{\cos t}}\cdot 4\,\frac{\sin t}{\cos^2 t}\,dt$$

$$=\underbrace{\int 16\,\frac{\sin^2 t}{\cos t}\,dt+\int 32\,\frac{\sin t}{\cos t}\,dt+\int\frac{5}{\cos t}\,dt}_{\text{Çözümü biraz kolay integral!}}$$

$\sqrt{a^2+x^2}$ veya $\sqrt{x^2+a^2}$

$$x = a \tan t \qquad dx = \frac{a}{\cos^2 t} dt$$

$$\int \frac{3x^2+4}{\sqrt{x^2+16}} dx = ? \qquad x = 4 \tan t \qquad dx = \frac{4}{\cos^2 t} dt$$

$$\int \frac{3(\tan t)^2+4}{\sqrt{(4 \tan t)^2+16}} \cdot \frac{4}{\cos^2 t} dt$$

$$\sqrt{16 \tan^2 t+16} = \sqrt{16\left(\frac{\sin^2 t}{\cos^2 t}+\frac{\cos^2 t}{\cos^2 t}\right)} = \frac{4}{\cos t}$$

$$= \int \frac{3 \tan^2 t+4}{\frac{4}{\cos t}} \frac{4}{\cos^2 t} dt = \int 3 \frac{\tan^2 t}{\cos t} dt + \int \frac{4}{\cos t} dt$$

çözülebilecek integral.

$\sqrt{ax+b}$ nin değişik kuvvetleri varsa

$(ax+b) = t^n$ şeklinde değişken

seçilir n bütün kökleri

kaldıracak şekilde seçilir.

$$\int \frac{\sqrt[3]{ax+b} + \sqrt[5]{ax+b} + 2}{\sqrt[2]{ax+b} + \sqrt[4]{ax+b} + 3}\, dx$$

$$\int \frac{(ax+b)^{1/3} + (ax+b)^{1/5} + 2}{(ax+b)^{\frac{1}{2}} + (ax+b)^{1/4} + 3}\, dx$$

3, 5, 2, 4 ü bölen en küçük 60 dır

$ax+b = t^{60}$ $\quad a\,dx = 60\, t^{59} dt$

$$= \int \frac{(t^{60})^{1/3} + (t^{60})^{\frac{1}{5}} + 2}{(t^{60})^{\frac{1}{2}} + (t^{60})^{\frac{1}{4}} + 3}\, 60\, t^{59}\, dt$$

$$= \int \frac{t^{20} + t^{12} + 2}{t^{30} + t^{15} + 3} \cdot 60 \cdot t^{59}\, dt$$

köklü ifade kalmadı.

çözebiliriz. (uzun zaman)

$$\int \frac{\sin t + \cos^2 t \tan t}{\cos^3 t + \sin^2 t \cos t}\, dt$$

Eğer sin cos tan vs rasyonel fonksiyonu ise

$$\tan\frac{x}{2} = t \rightarrow \sin x = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$\frac{x}{2} = \tan t \rightarrow \frac{dx}{2} = \frac{1}{1+t^2}\, dt \qquad dx = \frac{2}{1+t^2}\, dt$$

$$\tan x = \frac{2t}{1-t^2}$$

$$= \int \frac{\frac{2t}{1+t^2} + \left(\frac{1-t^2}{1+t^2}\right)^2 \left(\frac{2t}{1-t^2}\right)}{\left(\frac{1-t^2}{1+t^2}\right) + \left(\frac{2t}{1+t^2}\right)^2 \left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2}{1+t^2}\, dt$$

$$= \int \underbrace{\frac{t^6 + 2611 t^5 + \cdots}{t^7 + 3651 t^6 + 261 t^5 + \cdots}\, dt}_{\text{çözülebilir integral.}}$$

$\int \cos^n x \, dx$ integralini çözün.
(bir indirgeme bağıntısı bulun)

$$\int \cos^n x \, dx = \int \underbrace{\cos^{n-1} x}_{u} . \underbrace{\cos x \cdot dx}_{dv} = \varphi$$

$$u = \cos^{n-1} x \longrightarrow du = (n-1)\cos^{n-2} x \, (-\sin x) dx$$

$$v = \sin x \longleftarrow dv = \cos x . dx$$

$$\int u \, dv = uv - \int v \, du$$

$$\varphi = uv - \int v . du$$

$$\varphi = \cos^{n-1} x . \sin x - \int \sin x \, (n-1) \cos^{n-2} x \, (-\sin x) dx$$

$$= \cos^{n-1} x . \sin x + (n-1) \int \cos^{n-2} x \, \sin^2 x \, dx$$

$$= \cos^{n-1} x . \sin x + (n-1) \int \cos^{n-2} x \, (1 - \cos^2 x) \, dx$$

$$= \qquad + (n-1) \int (\cos^{n-2} x - \cos^{n-2} x . \cos^2 x) \, dx$$

$$\varphi = \cos^{n-1} x \, \sin x + (n-1) . \int \cos^{n-2} dx - (n-1) \underbrace{\int \cos^n x \, dx}_{\varphi}$$

φ lı terimi sol tarafa at

$$\varphi + (n-1)\varphi = \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x\,dx$$

$$\varphi[1-n-1] = \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x\,dx$$

$$\varphi = \frac{1}{n}\cos^{n-1}x \sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

$$\boxed{\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x \sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx}$$

Sınav sorusu adayı:

$$\varphi = \int \sin^7 x\,dx = A\cdot \sin^6 x \cos x + B\int \sin^5 x\,dx$$

indirgeme formülünü türetin A ve B'yi hesaplayın.

Cevap:

$$\varphi = \int \sin^7 x\,dx = \int \sin^6 x \cdot \sin x\,dx$$

$$u = \sin^6 x \rightarrow du = 6\sin^5 x \cos x$$

$$v = -\cos x \leftarrow dv = \sin x\,dx$$

$$\varphi = uv - \int v\, du$$ 3

$$\varphi = \sin^6 x \cdot (-\cos x) - \int (-\cos x)\, 6 \sin^5 x \cos x\, dx$$

$$\varphi = -\sin^6 x \cos x + 6 \int \sin^5 x \cdot \cos^2 x\, dx$$

$$+ 6 \int \sin^5 x (1 - \sin^2 x)\, dx$$

$$+ 6 \left[\int \sin^5 x\, dx - \int \sin^7 x\, dx \right]$$

$$\varphi = -\sin^6 x \cos x + 6 \int \sin^5 x\, dx - 6 \underbrace{\int \sin^7 x\, dx}_{\varphi}$$

φ yu sola at

$$\varphi + 6\varphi = -\sin^6 x \cos x + 6 \int \sin^5 x\, dx$$

$$7\varphi = \ldots$$

$$\varphi = -\frac{1}{7} \sin^6 x \cos x + \frac{6}{7} \int \sin^5 x\, dx$$

$$\int \sin^7 x\, dx = -\frac{1}{7} \sin^6 x \cos x + \frac{6}{7} \int \sin^5 x\, dx$$

$$\boxed{A = -\frac{1}{7}} \qquad \boxed{B = \frac{6}{7}}$$

Sınav Sorusu adayı

(25) $\int \cos^{10}x\,dx = A\cos^9 x \sin x + B\int \cos^8 x\,dx$

şeklinde indirgeme işlemini gerçekleştirin A ve B'yi hesaplayın

Cevap:

$$\varphi = \int \cos^{10}x\,dx = \int \underbrace{\cos^9 x}_{u} . \underbrace{\cos x . dx}_{v}$$

$$u = \cos^9 x \longrightarrow du = 9\cos^8 x . (-\sin x)$$

$$v = \sin x \longleftarrow dv = \cos x\,dx$$

$$\varphi = uv - \int v\,du$$

$$=$$

Bu sorudan sıfır almanın yolu

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x \sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

formulunde $n=10$ koy

$$\int \cos^{10}x\,dx = \frac{1}{10}\cos^9 x \sin x + \frac{9}{10}\int \cos^8 x\,dx$$

(26)

$\varphi = \int \frac{dx}{\sin^7 x} = \int \underbrace{\frac{1}{\sin^5 x}}_{u} \cdot \underbrace{\frac{dx}{\sin^2 x}}_{dv}$ şeklinde

tanım yaparak integrali indirgeyin.

<u>Cevap</u>

$u = \sin^{-5} x \longrightarrow du = -5\sin^{-6} x \cdot \cos x$.

$v = -\cot x \longleftarrow dv = \frac{dx}{\sin^2 x}$

$\varphi = uv - \int v\,du$

$\varphi = \sin^{-5} x\,(-\cot x) - \int (-\cot x)(-5\sin^{-6} x \cos x)$

$= -\sin^{-5} x \cot x - 5\int \cot x\, \sin^{-6} x \cdot \cos x \; dx$

$-5\int \frac{\cos x}{\sin x} \cdot \sin^{-6} x \cdot \cos x \; dx$

$-5\int \cos^2 x \cdot \sin^{-7} x \; dx$

$-5\int (1 - \sin^2 x)(\sin^{-7} x)\, dx$

$-5\int (\sin^{-7} x\, dx - \sin^{-5} x)\, dx$

$$\varphi = -\sin^{-5}x \cdot \cot x - 5\underbrace{\int \sin^{-7}x\,dx}_{\varphi} + 5\int \sin^{-5}x\,dx$$

$$\varphi + 5\varphi = -\sin^{-5}x \cdot \cot x + 5\int \sin^{-5}x\,dx$$

$$6\varphi = \cdots$$

$$\varphi = \frac{1}{6}\Big[\; \cdot \quad - \;$$

$$\int \frac{dx}{\sin^7 x} = -\frac{1}{6}\sin^{-5}x \cot x + \frac{5}{6}\int \sin^{-5}x\,dx$$

$$= -\frac{1}{6}\sin^{-5}x\,\frac{\cos x}{\sin x} + \frac{5}{6}\int \sin^{-5}x\,dx$$

$$\boxed{\int \frac{dx}{\sin^7 x} = -\frac{1}{6}\,\frac{\cos x}{\sin^6 x} + \frac{5}{6}\int \frac{dx}{\sin^5 x}}$$

Bu sorudan sıfır almanın yolu hazır formülü bir şekilde yazıp onu kullanmak.

221

$$\varphi = \int \tan^p x\,(1+\tan^2 x)\,dx = ?$$

Hatırlatma $(\tan x)' = (1+\tan^2 x)$

$u = \tan x \qquad du = (1+\tan^2 x)\,dx$

$$\varphi = \int u^p\,du = \frac{1}{p+1}\,u^{p+1} = \frac{1}{p+1}\tan^{p+1} x$$

222)

$\varphi = \int \tan^n x\,dx =$ Bir indirgeme formulu bulu

$$\int \tan^{n-2} x \cdot \tan^2 x\,dx$$

$$\int \tan^{n-2} x\,(1+\tan^2 x - 1)\,dx$$

$$\varphi = \int \tan^{n-2} x\,(1+\tan^2 x)\,dx - \int \tan^{n-2} x\,dx$$

yukarıdaki formulde $\int \tan^p x\,(1+\tan^2 x) = \dfrac{\tan^{p+1} x}{p+1}$

$$\int \tan^{n-2} x\,(1+\tan^2 x)\,dx = \frac{\tan^{n-2+1}}{n-2+1} \Rightarrow \frac{\tan^{n-1}}{n-1}$$

$$\varphi = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$$

$$\boxed{\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx}$$

$$\varphi \doteq \int \frac{dx}{(a^2+x^2)^n} = \int (a^2+x^2)^{-n} dx$$

$$u = (a^2+x^2)^{-n} \longrightarrow du = -n(a^2+x^2)^{-n-1} \cdot 2x$$

$$v = x \longleftarrow dv = dx$$

$$\varphi = uv - \int v \, du$$

$$= (a^2+x^2)^{-n} \cdot x - \int x \cdot (-n)(a^2+x^2)^{-n-1} \cdot 2x \, dx$$

$$+ 2n \underbrace{\int x^2 (a^2+x^2)^{-n-1} dx}_{P}$$

$$P = \int x^2 (a^2+x^2)^{-n-1} dx = \int (x^2+a^2-a^2)(a^2+x^2)^{-n-1} dx$$

$$= \int [(x^2+a^2) - a^2](a^2+x^2)^{n-1} dx$$

$$= \int (x^2+a^2)(a^2+x^2)^{-n-1} dx - a^2 \int (a^2+x^2)^{-n-1} dx$$

$$= \int (x^2+a^2)^{-n} dx - a^2 \int (a^2+x^2)^{-n-1} dx$$

Bu P yi φ ifadesinde yerine koy

$$\varphi = (a^2+x^2)^{-n}x + 2nP$$

$$=(a^2+x^2)^{-n}x + 2n\left[\int (x^2+a^2)^{-n}dx - a^2\int (a^2+x^2)^{-n-1}dx\right]$$

$$\varphi = \int \frac{dx}{(a^2+x^2)^n}$$

$$\int \frac{dx}{(a^2+x^2)^n} = \frac{x}{(a^2+x^2)^n} + 2n\int \frac{dx}{(a^2+x^2)^n} - 2na^2 \underbrace{\int \frac{dx}{(a^2+x^2)^{n+1}}}_{M}$$

M yi yalnız bırakalım.

$$\int \frac{dx}{(a^2+x^2)^n} - \frac{x}{(a^2+x^2)^n} - 2n\int \frac{dx}{(a^2+x^2)^n} = -2na^2\int \frac{dx}{(a^2+x^2)^{n+1}}$$

$$(1-2n)\int \frac{dx}{(a^2+x^2)^n} - \frac{x}{(a^2+x^2)^n} = -2na^2 \underbrace{\int \frac{dx}{(a^2+x^2)^{n+1}}}_{R}$$

R yi sol tarafa at ve yalnız bırak

$$\int \frac{dx}{(a^2+x^2)^{n+1}} = \frac{1}{2na^2}\,\frac{x}{(a^2+x^2)^n} - \frac{(1-2n)}{2na^2}\int \frac{dx}{(a^2+x^2)^n}$$

$$+\frac{2n-1}{2na^2}$$

veya $n \longrightarrow p-1$

$$\int \frac{dx}{(a^2+x^2)^p} = \frac{1}{2(p-1)a^2}\,\frac{x}{(a^2+x^2)^{p-1}} + \frac{2p-3}{2(p-1)a^2}\int \frac{dx}{(a^2+x^2)^{p-1}}$$

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226) yukarıdaki formulu kullanarak

$$\int \frac{dx}{(1+x^2)^2} = ?$$

Cevap

$$\int \frac{dx}{(1+x^2)^2} = \frac{1}{2(2-1)\cdot 1^2}\,\frac{x}{(1+x^2)^{2-1}} + \frac{2\cdot 2-3}{2(2-1)1}\int \frac{dx}{(1+x^2)}$$

$$= \frac{x}{2(1+x^2)} + \frac{1}{2}\int \frac{dx}{1+x^2}$$

$$\int \frac{dx}{(1+x^2)^2} = \frac{x}{2(1+x^2)} + \frac{1}{2}\,\mathrm{arc\ tan}\,x$$

$$\varphi = \int \frac{\ln 2x}{\ln 4x} \frac{dx}{x} = ?$$

$$\ln 4x = \ln(2 \cdot 2x) = \ln 2 + \ln 2x$$
$$= a + \ln 2x$$

$$\varphi = \int \frac{\ln 2x}{a + \ln 2x} \frac{dx}{x}$$

$$u = \ln 2x \rightarrow du = \frac{2}{2x} dx = \frac{dx}{x}$$

$$\varphi = \int \frac{u}{a+u} du$$

$$\begin{array}{r|l} u & u+a \\ \cline{2-2} -\ u+a & 1 \\ \cline{1-1} -a & \end{array}$$

$$\frac{u}{u+a} = 1 + \frac{-a}{u+a}$$

$$\varphi = \int \left(1 - \frac{a}{u+a}\right) du = \int du - a \int \frac{du}{u+a}$$

$$= u - a \cdot \ln(u+a)$$

$$\boxed{\varphi = \ln 2x - \ln 2 \ln(\ln 2x + \ln 2)}$$