

Use of Residues: Partial Fraction Expansion.

$$f(z) = \frac{p(z)}{q(z)} = \frac{A}{z-a} + \frac{B}{z-b} + \frac{C}{z-c} + \dots$$

First residue Formula

$$A = \text{Res}(z=a) = \lim_{z \rightarrow a} (z-a) \frac{p(z)}{q(z)}$$

$$B = \text{Res}(z=b) = \lim_{z \rightarrow b} (z-b) \frac{p(z)}{q(z)}$$

$$C = \text{Res}(z=c) = \lim_{z \rightarrow c} (z-c) \frac{p(z)}{q(z)}$$

Second residue Formula

$$A = \text{Res}(z=a) = \lim_{z \rightarrow a} \frac{p(z)}{q'(z)}$$

$$B = \text{Res}(z=b) = \lim_{z \rightarrow b} \frac{p(z)}{q'(z)}$$

$$C = \text{Res}(z=c) = \lim_{z \rightarrow c} \frac{p(z)}{q'(z)}$$

Example AE-611

$$\frac{z+5}{z^3-5z^2-2z+24} = \frac{z+5}{(z-4)(z+2)(z-3)} = \frac{A}{z-4} + \frac{B}{z+2} + \frac{C}{z-3}$$

$$A = \lim_{z \rightarrow 4} (z-4) \frac{p(z)}{q(z)} = \lim_{z \rightarrow 4} (z-4) \frac{z+5}{(z-4)(z-3)(z+2)} = \lim_{z \rightarrow 4} \frac{z+5}{(z-3)(z+2)} = \frac{4+5}{(4-3)(4+2)} = \frac{9}{6} = 1.5$$

$$B = \lim_{z \rightarrow -2} (z-(-2)) \frac{p(z)}{q(z)} = \lim_{z \rightarrow -2} (z+2) \frac{z+5}{(z-4)(z-3)(z+2)} = \lim_{z \rightarrow -2} \frac{z+5}{(z-4)(z-3)} = \frac{-2+5}{(-2-3)(-2-4)} = \frac{3}{30} = 0.1$$

$$C = \lim_{z \rightarrow 3} (z-3) \frac{p(z)}{q(z)} = \lim_{z \rightarrow 3} \frac{z+5}{(z-4)(z+2)} = \frac{3+5}{(3-4)(3+2)} = -1.6$$

Using the Second residue Formula

In our problem $p(z)=z+5$ $q(z)=z^3-5z^2-2z+24$ and $q'(z)=3z^2-10z-2$

$$A = \lim_{z \rightarrow 4} \frac{z+5}{3z^2-10z-2} = \frac{4+5}{3 \cdot 4^2-10 \cdot 4-2} = \frac{9}{6} = 1.5$$

$$B = \lim_{z \rightarrow -2} \frac{z+5}{3z^2-10z-2} = \frac{-2+5}{3 \cdot (-2)^2-10 \cdot (-2)-2} = \frac{3}{30} = 0.1$$

$$C = \lim_{z \rightarrow 3} \frac{z+5}{3z^2-10z-2} = \frac{3+5}{3 \cdot (3)^2-10 \cdot (3)-2} = \frac{8}{-5} = -1.6$$

Note we get the same result by **classical method**

$$\frac{z+5}{z^3-5z^2-2z+24} = \frac{A}{z-4} + \frac{B}{z+2} + \frac{C}{z-3}$$

$$= \frac{A(z+2)(z-3) + B(z-4)(z-3) + C(z-4)(z+2)}{(z-4)(z+2)(z-3)}$$

$$= \frac{A(z^2-z-6) + B(z^2-7z-12) + C(z^2-2z-8)}{(z-4)(z+2)(z-3)}$$

$$= \frac{z^2(A+B+C) + z(-7A-B-2C) + 12A-6B-8C}{(z-4)(z+2)(z-3)}$$

$$z^2(A+B+C) + z(-7A-B-2C) + 12A-6B-8C = z+5$$

$$A+B+C=0, \quad -7A-B-2C=1, \quad 12A-6B-8C=5$$

Solving for A,B,C we get $A=1.5$ $B=0.1$ $C=-1.6$

$$\frac{z+5}{z^3-5z^2-2z+24} = \frac{1.5}{z-4} + \frac{0.1}{z+2} + \frac{-1.6}{z-3}$$

Example AE-612

$$\frac{2z+12}{z^2+2z+2} = \frac{2z+12}{[z-(-1+i)][z-(-1-i)]} = \frac{2z+12}{(z+1-i)(z+1+i)}$$

$$= \frac{A}{(z+1-i)} + \frac{B}{(z+1+i)}$$

Using the **first residue formula**

$$A = \lim_{z \rightarrow -1+i} (z+1-i) \frac{2z+12}{(z+1-i)(z+1+i)} = \lim_{z \rightarrow -1+i} \frac{2z+12}{(z+1+i)}$$

$$= \frac{2(-1+i)+12}{((-1+i)+1+i)} = \frac{2i+10}{2i} = \frac{2i}{2i} + \frac{10}{2i} = 1-5i$$

$$B = \lim_{z \rightarrow -1-i} (z+1+i) \frac{2z+12}{(z+1-i)(z+1+i)} = \lim_{z \rightarrow -1-i} \frac{2z+12}{(z+1-i)} = 1+5i$$

Using the **second residue formula**

$$p(z)=2z+12, \quad q(z)=z^2+2z+2, \quad q'(z)=2z+2$$

$$A = \lim_{z \rightarrow -1+i} \frac{p(z)}{q'(z)} = \frac{2z+12}{2z+2} = \frac{2(-1+i)+12}{2(-1+i)+2} = 1-5i$$

$$B = \lim_{z \rightarrow -1-i} \frac{p(z)}{q'(z)} = \frac{2z+12}{2z+2} = \frac{2(-1-i)+12}{2(-1-i)+2} = 1+5i$$

Using **classical Method**

$$\frac{2z+12}{z^2+2z+2} = \frac{A}{(z+1-i)} + \frac{B}{(z+1+i)} = \frac{A(z+1+i) + B(z+1-i)}{(z+1-i)(z+1+i)}$$

$$(A+B)=2 \quad A(1+i)+B(1-i)=12 \quad \text{solution is } A=1-5i, \quad B=1+5i$$

Result
$$\frac{2z+12}{z^2+2z+2} = \frac{1-5i}{(z+1-i)} + \frac{1+5i}{(z+1+i)}$$