MATRIX EIGENVALUES

Problem AE-721. Find the eigenvalues and eigenvectors of the folllowing matrix.

$$A=\begin{bmatrix} 4 & 1 & 0 \\ -3 & -2 & 0 \\ 3 & 1 & 5 \end{bmatrix}$$

Solution:

$$A-\lambda I=\begin{bmatrix} 4 & 1 & 0 \\ -3 & -2 & 0 \\ 3 & 1 & 5 \end{bmatrix}-\lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 4-\lambda & 1 & 0 \\ -3 & -2-\lambda & 0 \\ 3 & 1 & 5-\lambda \end{bmatrix}$$

$$\text{Det}(A-\lambda I)=(5-\lambda)\begin{vmatrix} 4-\lambda & 1 \\ -3 & -2-\lambda \end{vmatrix}$$

$$=(5-\lambda)\left((4-\lambda)(-2-\lambda)+3\right)$$

$$=(5-\lambda)(-8-4\lambda+2\lambda+\lambda^2+3)=(5-\lambda)(\lambda^2-2\lambda-5)$$

$$=-\lambda^3+7\lambda^2-5\lambda-25$$

The roots are $\lambda_1=3.45$, $\lambda_2=-1.45$, $\lambda_3=5$

Thus the eigenvalues are **$\lambda_1=3.45$, $\lambda_2=-1.45$, $\lambda_3=5$**

Now we shall find the eigenvectors.

First we calculate eigenvector belong to $\lambda_1=3.45$.

Set $\lambda_1=3.45$ in $[A-\lambda I]X=0$ and calculate the vector X.

$$\begin{bmatrix} 4-\lambda_1 & 1 & 0 \\ -3 & -2-\lambda_1 & 0 \\ 3 & 1 & 5-\lambda_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4-3.45 & 1 & 0 \\ -3 & -2-3.45 & 0 \\ 3 & 1 & 5-3.45 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0.55 & 1 & 0 \\ -3 & -5.45 & 0 \\ 3 & 1 & 1.55 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\rightarrow \begin{matrix} 0.55\,x1+x2=0, \\ -3\,x1-5.45\,x2=0, \\ 3x1+x2+1.55x3=0, \end{matrix}$$

Set one variable free. Set $x_1=1$

$0.55\,x_1+x_2=0;\ \rightarrow\ 0.55\ 1+x_2=0;\ \rightarrow\ x_2=-0.55$

$-3\,x_1-5.45\,x_2=0;\rightarrow x_2=-0.55$

$3x_1+x_2+1.55\,x_3=0;\ \rightarrow\ 3\ 1+(-0.55)+1.55x_3=0;$

$x_3=\frac{-2.55}{1.55}=-1.645$

the eigenvector belong to the eigenvalue $\lambda_1=3.45$is

$$X_1=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 1 \\ -0.55 \\ -1.65 \end{bmatrix}$$

Now we calculate the **eigenvector belong to $\lambda_2=-1.45$.** Set $\lambda=-1.45$ in the equation $[A-\lambda I]X=0$ and calculate the vector X.

$$\begin{bmatrix} 4+1.45 & 1 & 0 \\ -3 & -2+1.45 & 0 \\ 3 & 1 & 5+1.45 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$5.55\,x_1+x_2=0;$

$-3\,x_1-0.55\,x_2=0;$

$3x_1+x_2+6.55x_3=0;$

Set $x_1=1$, calculate x_2

$5.55\,x_1+x_2=0;\ x_2=-5.55;$

$3x_1+x_2+6.55x_3=0;\ \rightarrow\ 3\ 1+(-5.55)+6.55x_3=0;$

$x_3=0.39$

The eigenvector belong to the eigenvalue $\lambda_1=-1.45$ is

$$X_2=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 1 \\ -5.55 \\ 0.39 \end{bmatrix}$$

Now we calculate the eigenvector belong to $\lambda_3=5$

Set $\lambda=5$ in the equation $[A-\lambda I]X=0$ and calculate the vector X.

$$\begin{bmatrix} 4-5 & 1 & 0 \\ -3 & -2-5 & 0 \\ 3 & 1 & 5-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ -3 & -7 & 0 \\ 3 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\rightarrow \begin{matrix} -x1+x2=0, \\ -3\,x1-7\,x2=0, \\ 3x1+x2=0, \end{matrix}$$

The coefficient of x_3 is zero in all three equations.

Any value of x_3 satisfy the equations.

Now we want to find the value of x_1 and x_2.

Set $x_1=1$ and calculate x_2

$-x_1+x_2=0;\quad \rightarrow -1+x_2=0,\ \rightarrow x_2=1$

However $x_1=1$, $x_2=1$ does not satisfy the second and third equation.

$-3\,x_1-0.55\,x_2=0;\ \rightarrow -3\ 1-0.55\ 1=0;\ \rightarrow -4.55=0$

$3x_1+x_2=0;\quad \rightarrow 3+1=0\quad \rightarrow\ 4=0$

The conflict can only be solved if we set $x_1=0$ and $x_2=0$. Thus the eigenvector belong to $\lambda_3=5$ is

$$X_3=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ \text{free} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We can set free variable x_3 any value. Here we set $x_3=1$. Notice: If we set $x_3=0$ then we lead to the trivial solution [0 0 0]. That was not our goal.

Comments: if X is an eigenvector, then αX is also an eigenvector.

Thus we have found that

$$X_1 = \begin{bmatrix} 1 \\ -0.55 \\ -1.65 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 \\ -5.55 \\ 0.39 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are eigenvectors. So $10X_1$, $100X_2$, $7X_3$ are also eigenvectors.

$$X_1 = \begin{bmatrix} 10 \\ -5.5 \\ -16.5 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 100 \\ -555 \\ 39 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}$$

Problem AE-722. Find the eigenvalues and eigenvectors of the folllowing matrix.

$$A = \begin{bmatrix} 4 & 2 \\ -1 & 6 \end{bmatrix}$$

Solution:

$$A-\lambda I = \begin{bmatrix} 4 & 2 \\ -1 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4-\lambda & 2 \\ -1 & 6-\lambda \end{bmatrix}$$

Det $(A-\lambda I) = (4-\lambda)(6-\lambda)-(-2) = \lambda 2-10\lambda+26$

The roots are $\lambda_1=5+i$, $\lambda_2=5-i$

Find the eigenvector belong to $\lambda_1=5+i$

$$\begin{bmatrix} 4-\lambda_1 & 2 \\ -1 & 6-\lambda_1 \end{bmatrix} = \begin{bmatrix} 4-(5+i) & 2 \\ -1 & 6-(5+i) \end{bmatrix} = \begin{bmatrix} -1-i & 2 \\ -1 & 1-i \end{bmatrix}$$

$$\begin{bmatrix} -1-i & 2 \\ -1 & 1-i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} (-1-i)x_1 + 2x_2 = 0 \\ -x_1 + (1-i)x_2 = 0 \end{matrix}$$

Set $x_1=1$, We get $2x_2=1+i$, $\rightarrow$ $x_2 = \frac{2}{1+i} = 1-i$

Thus the eigenvector belong to $\lambda_1=5+i$, is

$$X_1 = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$$

Similarly the eigenvector belong to $\lambda_1=5-i$, is

$$X_1 = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$$

Similarly