The Use of Eigenvalues

Solution of linear differential equations

$$\frac{dq}{dt} = 3q \quad \rightarrow \quad q(t) = C\, e^{3t}$$

$$\frac{dq}{dt} = -5q \quad \rightarrow \quad q(t) = C\, e^{-5t}$$

$$\frac{dq}{dt} = aq \quad \rightarrow \quad q(t) = C\, e^{at}$$

$$\begin{aligned}\frac{dq_1}{dt} &= aq_1 + bq_2 \\ \frac{dq_2}{dt} &= cq_1 + dq_2\end{aligned} \quad \rightarrow \quad \begin{bmatrix}\frac{dq_1}{dt} \\ \frac{dq_2}{dt}\end{bmatrix} = \begin{bmatrix} a & b \\ c & d\end{bmatrix}\begin{bmatrix} q_1 \\ q_2\end{bmatrix}$$

$q_1(t) = ?$ $\qquad q_2(t) = ?$

In general

$$\begin{bmatrix}\frac{dq_1}{dt} \\ \frac{dq_2}{dt} \\ \ldots \\ \frac{dq_n}{dt}\end{bmatrix} = \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ .. \\ .. \\ q_n\end{bmatrix}$$

The solution is given by

$$\begin{bmatrix} q_1(t) \\ q_2(t) \\ .. \\ .. \\ q_n\end{bmatrix} = C_1\begin{bmatrix} \\ X_1 \\ \\ \end{bmatrix} e^{\lambda_1 t} + C_2\begin{bmatrix} \\ X_2 \\ \\ \end{bmatrix} e^{\lambda_2 t} + \ldots\ldots + C_n\begin{bmatrix} \\ X_n \\ \\ \end{bmatrix} e^{\lambda_n t}$$

where $\lambda_1, \lambda_2, \ldots \lambda_n$, are eigenvalues and $X_1, X_2, \ldots X_n$, are eigenvectors of the matrix A. $C_1, C_2, \ldots C_n$ are constants that is related to the initial conditions.

Example AE-782 Find the solution of the following differential equation.

$$\begin{bmatrix}\frac{dq_1}{dt} \\ \frac{dq_2}{dt} \\ \frac{dq_3}{dt} \\ \frac{dq_4}{dt}\end{bmatrix} = \begin{bmatrix} 3 & 7 & -12 & 7 \\ 4 & 3 & -30 & 21 \\ -0.5 & 0 & 3 & 0 \\ -1 & 0 & 7 & -1\end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_n\end{bmatrix}$$

Eigenvalues and eigenvectors of the matrix A is

$\lambda_1=7.47$, $\lambda_2=4.03$, $\lambda_3=-3.15$, $\lambda_4=-0.35$,

$$X_1 = \begin{bmatrix} -8.2 \\ -5.4 \\ 0.92 \\ 1.73\end{bmatrix}, \quad X_2 = \begin{bmatrix} 7 \\ 1.32 \\ -3.41 \\ -6.13\end{bmatrix}, \quad X_3 = \begin{bmatrix} -7.2 \\ 6.76 \\ -0.59 \\ -1.44\end{bmatrix}, X_4 = \begin{bmatrix} -9.48 \\ 2.77 \\ -1.42 \\ -0.66\end{bmatrix}$$

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4\end{bmatrix} = C_1\begin{bmatrix} -8.2 \\ -5.4 \\ 0.92 \\ 1.73\end{bmatrix} e^{7.4t} + C_2\begin{bmatrix} 7 \\ 1.32 \\ -3.41 \\ -6.13\end{bmatrix} e^{4t} + C_3\begin{bmatrix} -7.2 \\ 6.76 \\ -0.59 \\ -1.44\end{bmatrix} e^{-3.1t} + C_4\begin{bmatrix} -9.48 \\ 2.77 \\ -1.42 \\ -0.66\end{bmatrix} e^{-0.35t}$$

or

$$q_1(t) = -C_1 8.2e^{7.4t} + C_2 7e^{4t} - C_3 7.2e^{-3.1t} - C_4 9.48e^{-0.35t}$$

$$q_2(t) = -C_1 5.4e^{7.4t} + C_2 1.3e^{4t} + C_3 6.7e^{-3.1t} + C_4 2.7e^{-0.35t}$$

$q_3(t)=$

$q_4(t)=$

Example AE-783 Find the solution of the the differential equation $\frac{dq}{dt} = 4q$ with the initial condition q(0)=2;

Solution: $q(t)=Ce^{4t}$ replace t=0.

$q(0)=Ce^{4\,0}$

replace q(0)=2

$2=Ce^{0} \rightarrow C=2$

Thus the solution is $q(t)=2e^{4t}$

Example AE-784 Find the solution of the the differential equation $\frac{dq}{dt} = 5q$ with the initial condition q(1)=3;

Solution: $q(t)=Ce^{5t}$ replace t=1.

$q(1)=Ce^{5\,1}$

replace q(1)=3

$3=Ce^{5} \rightarrow C=3e^{-5}=3\ 0.0067= 0.0202$

Thus the solution is $q(t)=0.0202e^{5t}$

Example AE-785 Find the solution of the following differential equation.

$$\begin{bmatrix}\frac{dp}{dt} \\ \frac{dq}{dt}\end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 4 & 6\end{bmatrix}\begin{bmatrix} p \\ q\end{bmatrix}$$ with initial condition p(0)=60,

q(0)=20;

Solution $\lambda_1=2$, $\lambda_2=8$, $X_1=\begin{bmatrix}-7\\7\end{bmatrix}$, $X_2=\begin{bmatrix}-4.4\\9\end{bmatrix}$

$p(t)=-7C_1e^{2t} - 4.4C_2e^{8t}$
$q(t)=7C_1e^{2t} - 9C_2e^{8t}$
replace t=0
$p(0)=-7C_1e^{2\,0} - 4.4C_2e^{8\,0}$
$q(0)=7C_1e^{2\,0} - 9C_2e^{8\,0}$

$60=-7C_1 - 4.4C_2$
$20=7C_1 - 9C_2$

$C_1=-6.7 \quad C_2=-3$
Thus the required solution is
$p(t)= -7\ 6.7\ e^{2t} - 4.4(-3)\ e^{8t}=-46.9e^{2t} - 13.2\ e^{8t}$
$q(t)=7\ 6.7e^{2t} - 9(-3)\ e^{8t}=46.9e^{2t} - 27\ e^{8t}$

Example AE-786 Find the solution of the following differential equation.

$\begin{bmatrix}\frac{dp}{dt}\\ \frac{dq}{dt}\end{bmatrix}=\begin{bmatrix}4 & 2\\ -1 & 6\end{bmatrix}\begin{bmatrix}p\\ q\end{bmatrix}$ with initial condition p(0)=1, q(0)=2;

Solution:
Eigenvalues and eigenvectors are
$\lambda_1=5+i$, $\lambda_2=5-i$, $X_1=\begin{bmatrix}1\\1-i\end{bmatrix}$, $X_2=\begin{bmatrix}1\\1+i\end{bmatrix}$
Thus the solution is

$$p(t)=C_1\,1\,e^{(5+i)t}+C_2\,1\,e^{(5-i)t}$$
$$q(t)=C_1\,(1-i)\,e^{(5+i)t}+(1+i)C_2\ \ e^{(5-i)t}$$

Now find the constant coefficients C_1 and C_2.
It is given that p(0)=1 and q(0)=2. Thus replace t=0 in the solution equations

$$p(0)=C_1\,1\,e^{(5+i)0}+C_2\,1\,e^{(5-i)0} \quad \text{or}$$
$$1=C_1+C_2$$

$$q(0)=C_1\,(1-i)\,e^{(5+i)0}+(1+i)C_2\ \ e^{(5-i)0} \quad \text{or}$$
$$2=C_1\,(1-i)+(1+i)C_2$$

From the first equation $C_1=1-C_2$
Substitute this C_1 value into second equation

$$2=(1-C_2)\,(1-i)+(1+i)C_2$$
$$2=(1-i)-(1-i)C_2\ +(1+i)C_2$$

$2=(1-i)+2i\ C_2 \ \rightarrow\ C_2=\frac{1+i}{2i}=0.5-0.5i$
$C_1=1-C_2=1-(0.5-0.5i)=0.5+0.5i$
Thus the solutions is

$$p(t)=(0.5+0.5i)\,e^{(5+i)t}+(0.5-0.5i)\,1\,e^{(5-i)t}$$
$$q(t)=(0.5+0.5i)\,(1-i)\,e^{(5+i)t}+(0.5-0.5i)(1+i)e^{(5-i)t}$$

Note

$$p(t)=(0.5+0.5i)\,e^{(5+i)t}+(0.5-0.5i)\,1\,e^{(5-i)t}$$
$$=(0.5+0.5i)\,e^{5t}e^{it}+(0.5-0.5i)\,e^{5t}e^{-it}$$
$$=e^{5t}\left((0.5+0.5i)\,e^{it}+(0.5-0.5i)\,1\,e^{-it}\right)$$
$$=e^{5t}\left(0.5e^{it}+0.5e^{-it}+0.5i\,e^{it}-0.5i\,e^{-it}\right)$$
$$=e^{5t}\left(0.5(e^{it}+e^{-it})+0.5i\,(e^{it}-e^{-it})\right)$$

replace $\frac{e^{it}+e^{-it}}{2}=\cos t \qquad \frac{e^{it}-e^{-it}}{2i}=\sin t$

$$=e^{5t}\left(0.5(2\cos t)+0.5i\,(2i\sin t)\right)$$
$$=e^{5t}\left(\cos t-\sin t\right)$$

In a similar manner

$$q(t)=(0.5+0.5i)\,(1-i)\,e^{(5+i)t}+(0.5-0.5i)(1+i)e^{(5-i)t}$$
$$=e^{(5+i)t}+e^{(5-i)t}=e^{5t}(e^{it}+e^{-it})=2e^{5t}\cos t$$

Result the solution is
$p(t)=e^{5t}\left(\cos t-\sin t\right) \qquad q(t)=2e^{5t}\cos t$

Example AE-786 Find the solution of the following differential equation.

$$\begin{bmatrix}\frac{dp}{dt}\\ \frac{dq}{dt}\\ \frac{dr}{dt}\end{bmatrix}=\begin{bmatrix}1 & 0 & -3.1\\ 3 & 7 & 2\\ 3 & 0 & 2\end{bmatrix}\begin{bmatrix}p\\ q\\ r\end{bmatrix}$$

Solution: $\lambda_1=7$, $\lambda_1=1.5+3i$, $\lambda_2=1.5-3i$,

$X_1=\begin{bmatrix}0\\1\\0\end{bmatrix}$, $X_2=\begin{bmatrix}-6.58-1.09i\\3.5+0.13i\\6.5i\end{bmatrix}$, $X_3=\begin{bmatrix}-6.58+1.09i\\3.5-0.13i\\-6.5i\end{bmatrix}$,

$$p(t)=C_1e^{7t}+e^{1.5t}(A\cos 3t+B\sin 3t)$$
$$q(t)=C_2e^{7t}+e^{1.5t}(D\cos 3t+E\sin 3t)$$
$$r(t)=C_3e^{7t}+e^{1.5t}(F\cos 3t+G\sin 3t)$$