Application of th eigenvalues

Obtain the state equations for the following circuit

From the first loop $-V_S+R_1I_1+V_L+V_{C2}=0$ (1)
From the second loop $-V_{c2}+V_{c3}\, I_1+ R_3\, I_3 =0$ (2)
The node equation $I_1=I_2+I_3$ (3)

From (3) $I_3= I_1-I_2$
From (2) $-V_{c2} + V_{c3} + R_3 (I_1-I_2) = 0$

$-V_{c2} + V_{c3} + R_3 I_1 - R_3 I_2 = 0$

$-V_{c2} + V_{c3} + R_3 I_1 = R_3 I_2$

Now replace $I_2 = C \dfrac{dV_{c2}}{dt}$

$$-V_{c2} + V_{c3} + R_3 I_1 = R_3\, C \frac{dV_{c2}}{dt}$$

$$\frac{dV_{c2}}{dt} = -\frac{1}{R_3C_2}V_{c2} + \frac{1}{R_3C_2}V_{c3} + \frac{1}{R_3C_2}R_3 I_1 \quad \text{(E1)}$$

From 1 $V_L=V_s-R_1I_1-V_{c2}$

Replace $V_L= L\, \dfrac{dI_L}{dt} = \dfrac{dI_1}{dt}$

$$L\, \frac{dI_1}{dt} = V_s-R_1I_1-V_{c2}$$

$$\frac{dI_1}{dt} = \frac{1}{L} V_s - \frac{1}{L} R_1 I_1 - \frac{1}{L} V_{c2} \quad \text{(E2)}$$

From 2 $R_3 I_3 = V_{c2}-V_{c3}$

Replace $I_3= C_3 \dfrac{dv_{c3}}{dt}$

From 2 $R_3\, C_3 \dfrac{dv_{c3}}{dt} = V_{c2}-V_{c3}$

$$\frac{dv_{c3}}{dt} = \frac{1}{R_3\, C_3} V_{c2} - \frac{1}{R_3\, C_3} V_{c3} \quad \text{(E3)}$$

Set $V_s=0$ and Write (E1) (E2) (E3) in matrix form

$$\begin{bmatrix} \dfrac{dV_{c2}}{dt} \\ \dfrac{dI_1}{dt} \\ \dfrac{dV_{c3}}{dt} \end{bmatrix} = \begin{bmatrix} \dfrac{-1}{R_3C_2} & \dfrac{1}{C_2} & \dfrac{1}{R_3C_2} \\ \dfrac{-1}{L} & \dfrac{-R_1}{L} & 0 \\ \dfrac{1}{R_3C_3} & 0 & \dfrac{-1}{R_3C_3} \end{bmatrix} \begin{bmatrix} V_{c2} \\ I_1 \\ V_{c3} \end{bmatrix}$$

This is linear differential equations.

Problem AE-794 Calculate $V_{C2}(t)$, $I_1(t)$, $V_{C3}(t)$in the above circuit if $R_3=10$, $C_2=0.1$, $L=0.1$, $R_1=1$, $C_3=0.1$. Sketch the graph of $V_{C2}(t)$, $I_1(t)$, $V_{C3}(t)$ if $V_{C2}(0)=200V$, $I_1(0)=-10A$, $V_{C3}(0)=20V$
Solution Replace $R_3=10$, $C_2=0.1$, $L=0.1$, $R_1=1$, $C_3=0.1$ in the above equation.

$$\begin{bmatrix} \dfrac{dV_{c2}}{dt} \\ \dfrac{dI_1}{dt} \\ \dfrac{dV_{c3}}{dt} \end{bmatrix} = \begin{bmatrix} -1 & 10 & 1 \\ -10 & -10 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} V_{c2} \\ I_1 \\ V_{c3} \end{bmatrix}$$

Eigenvalues and eigenvectors of A matrix are
$\lambda_1=-0.9$, $\lambda_2=-5.54 + 8.89i$ $\lambda_2=-5.54 + 8.89i$

$$\mathbf{X_1}=\begin{bmatrix} 8.9 \\ -9.8 \\ 99 \end{bmatrix}, \mathbf{X_2}=\begin{bmatrix} 6.4-2.8i \\ -0.3+7i \\ -0.5-0.4i \end{bmatrix}, \mathbf{X_3}=\begin{bmatrix} 6.4+2.8i \\ -0.3-7i \\ -0.5-+0.4i \end{bmatrix}$$

Thus the solution of the circuit is

$$\begin{bmatrix} V_{C2} \\ I_L \\ V_{C3} \end{bmatrix} = C_1 \begin{bmatrix} 8.9 \\ -9.8 \\ 99 \end{bmatrix} e^{-0.9t} + C_2 \begin{bmatrix} 6.4-2.8i \\ -0.3+7.07 \\ -0.5-0.4i \end{bmatrix} e^{(-5.54+8.9i)t} + C_3 \begin{bmatrix} 6.4+2.8i \\ -0.3-7.07 \\ -0.5+0.4i \end{bmatrix} e^{(-5.54-8.9i)t}$$

or

$V_{C2}(t)=C_1\, 8.9\, e^{-0.9t} + C_2\, (6.4-2.8i)\, e^{(-5.54+8.9i)\, t}$
$+ C_3\, (6.4+2.8\, i)\, e^{(-5.54-8.9i)t}$

$I_L(t)=-C_1\, 9.8\, e^{-0.9t} +(-0.3+7.07i)\, C_2\, e^{(-5.54+8.9i)\, t}$
$+(-0.3-7.07i)\, C_3\, e^{(-5.54-8.9i)t}$

$V_{C3}(t)=C_1\, 99\, e^{-0.9t} + C_2\, (-0.5-0.4i)\, e^{(-5.54+8.9i)\, t}$
$+ C_3\, (-0.5+0.4i)\, e^{(-5.54-8.9)t}$

replace t=0

$200 = C_1\, 8.9 + C_2\,(6.4-2.8i) + C_3\,(6.4+2.8\,i)$

$-10 = -C_1\, 9.8 + C_2\,(-0.3+7.07i) + (-0.3-7.07i)\, C_3$

$20 = C_1\, 99 + C_2\,(-0.5-0.4i) + C_3\,(-0.5+0.4i)$

$$\begin{bmatrix} 8.9 & 6.4-2.8i & 6.4+2.8i \\ 9.8 & -0.3+7.07i & -0.3-7.07i \\ 99 & -0.5-0.4i & -0.5+0.4i \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 200 \\ -10 \\ 20 \end{bmatrix}$$

The solution is

$C_1 = 0.35$

$C_2 = 15.24 + 0.31i$

$C_3 = 15.24 - 0.31i$

Thus the solution is

$V_{C2}(t) = 3.11\, e^{-0.9t} + (98-40i)\, e^{(-5.54+8.9i)\,t}$
$\quad + (98+40i)\, e^{(-5.54-8.9i)t}$

$I_L(t) = -3.43\, e^{-0.9t} + (-6.7+107i)\, e^{(-5.54+8.9i)\,t}$
$\quad + (-6.7-107i)\, e^{(-5.54-8.9i)t}$

$V_{C3}(t) = 34.6e^{-0.9t} + (-7.5-6.25i)\, e^{(-5.54+8.9i)\,t}$
$\quad + (-7.5+6.25i)\, e^{(-5.54-8.9)t}$

Or if we write in sinusoid al form as in Example AE-782

$V_{C2}(t) = 3.11\, e^{-0.9t} + 196\cos(8.9t) + 80\sin(8.9t)$

$I_L(t) = -3.43\, e^{-0.9t} - 13.4\cos(8.9t) - 214\sin(8.9t)$

$V_{C3}(t) = 34.6e^{-0.9t} - 15\cos(8.9t) + 12.5\sin(8.9t)$