Analytical Methods in Engineering 3rd Major. 28.12.2003

1) The matrix A has the following eigenvectors. Calculate eigenvalues.

$$A=\begin{bmatrix} 3 & 2 & 4 \\ 3 & 4 & 2 \\ -2 & -2 & -1 \end{bmatrix}, \quad X_1=\begin{bmatrix} 2 \\ -5 \\ 2 \end{bmatrix} \quad X_2=\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad X_3=\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

Hint: Use the eigenvector formula $A\,X_i=\lambda_i\,X_i$

2) Find the eigenvalues and eigenvectors of the following matrices

(a) $A=\begin{bmatrix} 2 & -3 \\ 2 & 0 \end{bmatrix}$ (b) $B=\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$

3) Prove that eigenvalues of a triangular matrix are diagonal elements.

4) Solve the following matrix differential equation.

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix}=\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$ with the initial conditions $X_1(0)=1,\quad X_2(0)=2$

Eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$ are given as $\lambda_1=4,\quad \lambda_2=6$

$$X_1=\begin{bmatrix} 1 \\ -1 \end{bmatrix},\quad X_2=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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$$Ax_i = \begin{bmatrix} 3 & 2 & 4 \\ 3 & 4 & 2 \\ -2 & -2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 - 10 + 8 \\ 6 - 20 + 4 \\ -4 + 10 - 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \\ 4 \end{bmatrix}$$

$$Ax_i = \lambda_i x_i \qquad \begin{bmatrix} 4 \\ -10 \\ 4 \end{bmatrix} = \lambda_1 \begin{bmatrix} 2 \\ -5 \\ 2 \end{bmatrix} \qquad \boxed{\lambda_1 = 2}$$

$$\begin{bmatrix} 3 & 2 & 4 \\ 3 & 4 & 2 \\ -2 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 - 2 \\ 3 - 4 \\ -2 + 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \lambda_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \Rightarrow \lambda_2 = 1$$

$$\begin{bmatrix} 3 & 2 & 4 \\ 3 & 4 & 2 \\ -2 & -2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 + 4 \\ -8 + 2 \\ 4 - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -6 \\ 3 \end{bmatrix} = \lambda_3 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \Rightarrow \lambda = 3$$

(2)

$$A=\begin{bmatrix}2 & -3\\ 2 & 0\end{bmatrix} \qquad |A-\lambda I| = \begin{bmatrix}2-\lambda & -3\\ 2 & -\lambda\end{bmatrix}$$ (2)

$$-2\lambda+\lambda^2+6=0 \qquad \lambda^2-2\lambda+6=0$$

$$\lambda_{1,2}=\frac{2\mp\sqrt{4-24}}{2}=1\mp i\sqrt{5}$$

$$=1\mp i\,2.25$$

eigenvectors

$$\begin{bmatrix}2-\lambda & -3\\ 2 & -\lambda\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}2-1+i\,2.25 & -3\\ 2 & -1+2.25i\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}=\begin{matrix}0\\0\end{matrix}$$

$$(1+i\,2.25)a-3b=0 \qquad \text{set } b=1$$

$$a=\frac{3}{1+i\,2.25}=\frac{3(1-i\,2.25)}{1^2+2.25^2}$$

$$=\frac{3-6.75i}{6}=0.5-1.01i$$

$$X_1=\begin{bmatrix}0.5-1.01i\\ 1\end{bmatrix}$$

$$X_2=X_1^*=\begin{pmatrix}0.5+1.01i\\ 1\end{pmatrix}$$

$$B=\begin{Bmatrix}1 & 3\\ 0 & 2\end{Bmatrix} \qquad [B-\lambda I]=\begin{Bmatrix}1-\lambda & 3\\ 0 & 2-\lambda\end{Bmatrix}$$

$$(1-\lambda)(2-\lambda)=0$$

$$\lambda_1=1 \qquad \lambda_1=2$$

$$\begin{bmatrix}1-2 & 3\\ 0 & 2-2\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}\Rightarrow\begin{pmatrix}a=1\\ b=3\end{pmatrix}$$

$$\cancel{\begin{bmatrix}1-1 & 3\\ 0 & 2-1\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}\quad a=}$$

$$\begin{bmatrix} 1-1 & 3 \\ 0 & 2-1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$0a + 3b = 0 \Rightarrow b = 0$ (3)

$a = \text{free} = 1$

$\lambda_1 = 2 \qquad x_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\lambda_1 = 1 \qquad x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

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(4)

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} x_1 \end{bmatrix} e^{\lambda_1 t} + c_2 \begin{bmatrix} x_2 \end{bmatrix} e^{\lambda_2 t} \qquad = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t}$$

for $t = 0$

$$= c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} 1 + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} 1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 + c_2 \\ -c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$c_1 + c_2 = 1$

$-c_1 + c_2 = 2$

$c_1 = -0.5$

$c_2 = 1.5$

$x_1(t) = -0.5\, e^{4t} + 1.5\, e^{6t}$

$x_2(t) = 0.5\, e^{4t} + 1.5\, e^{6t}$
