EE300 Analytical Methods in Engineering 3rd Major. 15.5.2004

1)Calculate P and Q in the following procedure.

$$\frac{1.5z^3+8z^2+8z+2.5}{z^4+7z^3+9z^2-27z-54}=\frac{1.5z^3+8z^2+8z+2.5}{(z+3)^3(z-2)}=\frac{1}{(z+3)}+\frac{P}{(z+3)^2}+\frac{Q}{(z+3)^3}+\frac{0.5}{(z-2)}$$

2) Find the eigenvalues and eigenvectors of the following matrices

(a) $A=\begin{bmatrix}12 & -3\\ 8 & 2\end{bmatrix}$ b)$B=\begin{bmatrix}1 & 5\\ -4 & 5\end{bmatrix}$

3)The matrix A is given as $A=\begin{bmatrix}-5 & 6 & -3\\ 1 & 0 & 1\\ 8 & -8 & 6\end{bmatrix}$, Determine which of the following vectors could be eigenvectors of A. $X_1=\begin{bmatrix}0\\2\\4\end{bmatrix}$ $X_2=\begin{bmatrix}1\\0\\0\end{bmatrix}$ $X_3=\begin{bmatrix}2\\0\\2\end{bmatrix}$, $X_4=\begin{bmatrix}2\\0\\-2\end{bmatrix}$, $X_5=\begin{bmatrix}3\\3\\0\end{bmatrix}$

4)A Linear Matrix differential equation system is described by

$\begin{bmatrix}\frac{dx_1}{dt}\\ \frac{dx_2}{dt}\end{bmatrix}=[A]\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ and the eigenvalues and eigenvectors of A are $\lambda_1=2+3i$, $\lambda_2=2-3i$, $V_1=\begin{bmatrix}1\\ 5+i\end{bmatrix}$, $V_2=\begin{bmatrix}1\\ 5-i\end{bmatrix}$

Initial conditions are $x_1(0)=1$; $x_2(0)=0$; Calculate $x_1(t)$, and $x_2(t)$. Note: $x_1(t)$ and $x_2(t)$ should not contain complex variable i. Convert complex identities into sine and cosine terms. The final answer must be in the form of $x_1(t)=e^{\lambda t}(A\cos(wt)+B\sin(wt))$

5)Examine the following functions, $x_1(t)=e^{2t}$, $x_2(t)=e^{-2t}$, $x_3(t)=e^{2t}\cos(3t)$, $x_4(t)=e^{-2t}\cos(3t)$, $x_5(t)=e^{2t}+\cos(3t)$, $x_6(t)=e^{-2t}+\cos(3t)$

a) Plot the approximate graph of these functions.
b) Calculate the limit values $\lim_{t\to\infty} x_1(t)$, $\lim_{t\to\infty} x_2(t)$ .... $\lim_{t\to\infty} x_6(t)$

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1) $$\varphi = (z-3)^3 f(z)\Big|_{z=-3} = (z-3)^3 \frac{1.5z^3+8z^2+8z+2.5}{(z-3)^3(z-2)}\Bigg|_{z=-3}$$

$$= \frac{1.5z^3+8z^2+8z+2.5}{z-2}\Bigg|_{z=-3} =$$

$$= \frac{1.5(-3)^3+8(-3)^2+8(-3)+2.5}{-3-2} = \frac{10}{-5} = -2$$

$$P = \frac{d}{dz}\left[(z-3)^3 f(z)\right]\Bigg|_{z=-3} = \frac{d}{dz}\left[\frac{1.5z^3+8z^2+8z+2.5}{z-2}\right]\Bigg|_{z=-3}$$

$$= \frac{(1.5\times 3\times z^2+16z+8)(z-2)-(1.5z^3+8z^2+8z+2.5)}{(z-2)^2}\Bigg|_{z=-3} =$$

$$= \frac{0.5(-5)-10}{(-3-2)^2} = \frac{-12.5}{25} = -0.5$$

2) $\det[A-\lambda I] = \begin{vmatrix} 12-\lambda & -3 \\ 8 & 2-\lambda \end{vmatrix} = (12-\lambda)(2-\lambda) - 8\times(-3)$

$= \lambda^2 - 14\lambda + 48 = 0$

$\lambda_1 = 8 \quad \lambda_2 = 6$

$[A-\lambda I][x] = 0$

$$\begin{vmatrix} 12-\lambda & -3 \\ 8 & 2-\lambda \end{vmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\lambda_1 = 8$

$$\begin{bmatrix} 12-8 & -3 \\ 8 & 2-8 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4p - 3q = 0$ set $p=1$ $q = \frac{4}{3}$

The eigenvector of $\lambda_1 = 8$ is $X_1 = \begin{bmatrix} 1 \\ \frac{4}{3} \end{bmatrix}$

$\lambda_2 = 6$

$$\begin{vmatrix} 12-6 & -3 \\ 8 & 2-6 \end{vmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$6p - 3q = 0$

set $p = 1 \Rightarrow q = 2$

eigenvector for $\lambda_2 = 6$ is $X_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

③

$$B = \begin{bmatrix} 1 & 5 \\ -4 & 5 \end{bmatrix} \qquad \det\{B - \lambda I\} = \begin{vmatrix} 1-\lambda & 5 \\ -4 & 5-\lambda \end{vmatrix} =$$

$$= (1-\lambda)(5-\lambda) - (-4)5 = \lambda^2 - 6\lambda + 25 = 0$$

$$\lambda_1 = 3 + 4i$$

$$\lambda_2 = 3 - 4i$$

$$\begin{bmatrix} 1-\lambda & 5 \\ -4 & 5-\lambda \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\lambda_1 = 3 + 4i$

$$\begin{bmatrix} 1-(3+4i) & 5 \\ -4 & 5-(3-4i) \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\{1-(3+4i)\}\, p + 5q = 0$$

$$(-2-4i)\, p + 5q = 0$$

Set $p = 1 \Rightarrow q = -\dfrac{-2-4i}{5} = 0.4 + 0.8i$

eigenvector for $\lambda_1 = 3 + 4i$ is $X_1 = \begin{bmatrix} 1 \\ 0.4 + 0.8i \end{bmatrix}$

eigenvector for $\lambda_2 = 3 - 4i$ is $X_2 = \begin{bmatrix} 1 \\ 0.4 - 0.8i \end{bmatrix}$

Note: No need to solve for X_2

Because $\boxed{X_2 = \mathrm{conj}(X_1)}$

$$\frac{1}{0.4+0.8i} = \frac{0.4-0.8i}{(0.4+0.8i)(0.4-0.8i)} = \frac{0.4-0.8i}{0.4^2+0.8^2} =$$

$$= \frac{}{0.16+0.64} = \frac{0.4-0.8i}{0.80} = 0.5-i$$

$$\cancel{X_1 = 0.5-i}$$

$$X_1 = \begin{bmatrix} \dfrac{1}{0.4+0.8i} \\ \dfrac{0.4+0.8i}{0.4+0.8i} \end{bmatrix} = \begin{bmatrix} 0.5-i \\ 1 \end{bmatrix}$$

$$X_2 = \begin{pmatrix} 0.5+i \\ 1 \end{pmatrix}$$

3 $\begin{bmatrix} -5 & 6 & -3 \\ 1 & 0 & 1 \\ 8 & -8 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 8 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}}_{Ax_1} \overset{?}{=} \lambda \underbrace{\begin{bmatrix} 0 \\ 4 \\ 8 \end{bmatrix}}_{\lambda x_1}$$

Yes $\lambda = 2$ satisfies this equation

$A x_1 = \lambda x_1$

Thus $x_1 = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$ is an eigenvector of A.

$\begin{bmatrix} -5 & 6 & -3 \\ 1 & 0 & 1 \\ 8 & -8 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \\ 8 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} -5 \\ 1 \\ 8 \end{bmatrix}$ No λ satisfies this equation

$x_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ cannot be an eigenvector of A

$\begin{bmatrix} -5 & 6 & -3 \\ 1 & 0 & 1 \\ 8 & -8 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -16 \\ 4 \\ 28 \end{bmatrix}$ $x_3 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$ is not eigenvect.

$\begin{bmatrix} -5 & 6 & -3 \\ 1 & 0 & 1 \\ 8 & -8 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 4 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = \lambda \begin{bmatrix} -4 \\ 0 \\ 4 \end{bmatrix}$

Yes x_4 is eigenvector.

(5)

4) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 5+i \end{bmatrix} e^{(2+3i)t} + c_2 \begin{bmatrix} 1 \\ 5-i \end{bmatrix} e^{(2-3i)t}$

for $t=0$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 5+i \end{bmatrix} e^{0} + c_2 \begin{bmatrix} 1 \\ 5-i \end{bmatrix} e^{0}$$

$$\left(\begin{array}{l} c_1 + c_2 = 1 \\ (5+i)c_1 + (5-i)c_2 = 0 \end{array} \right) \Rightarrow \begin{array}{l} c_1 = 0.5 + 2.5i \\ c_2 = 0.5 - 2.5i \end{array}$$

$$x_1(t) = c_1 e^{(2+3i)t} + c_2 e^{(2-3i)t}$$

$$x_2(t) = c_1(5+i) e^{(2+3i)t} + c_2(5-i) e^{(2-3i)t}$$

$$x_1(t) = (0.5 + 2.5i) e^{(2+3i)t} + (0.5 - 2.5i) e^{(2-3i)t}$$

$$= e^{2t}\left[0.5 e^{3it} + 0.5 e^{-3it} + 2.5i e^{3it} - 2.5i e^{-3it}\right]$$

$$= e^{2t}\left[0.5\left(\frac{e^{3it} + e^{-3it}}{2} \times 2\right) + 2.5i\left(\frac{e^{3it} - e^{-3it}}{2i} \times 2i\right)\right]$$

$$= e^{2t}\left[0.5 \cos 3t \times 2 + 2.5i \sin 3t \times 2i\right]$$

$$\boxed{x_1(t) = e^{2t}\left\{\cos 3t - 5 \sin 3t\right\}}$$

$$x_2(t) = c_1 (5+i)\, e^{(2+3i)t} + c_2 (5-i)\, e^{(2-3i)t}$$

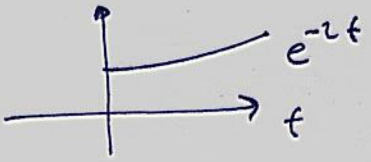
$$= e^{2t}\left[(0.5+2.5i)(5+i)\,e^{3it} + (0.5-2.5i)(5-i)\,e^{-3it}\right]$$

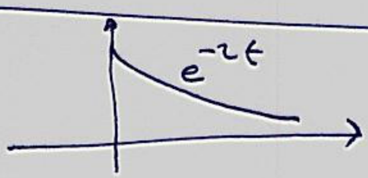
$$= e^{2t}\left[13i\, e^{3it} - 13i\, e^{-3it}\right]$$

$$= e^{2t}\left[13i\left(\frac{e^{3it} - e^{-3it}}{2i}\right) \times 2i\right]$$

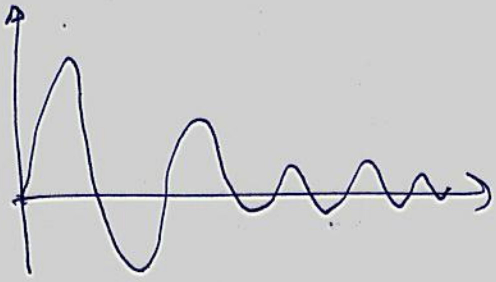
$$= e^{2t}\, 13i\, \sin 3t \times 2i$$

$$\boxed{x_2(t) = -e^{2t} \cdot 26 \sin 3t}$$

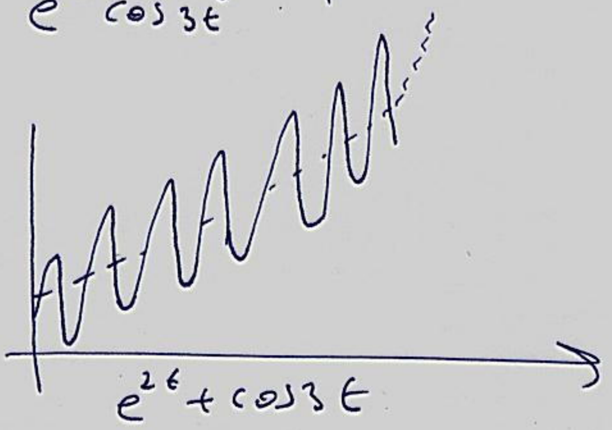




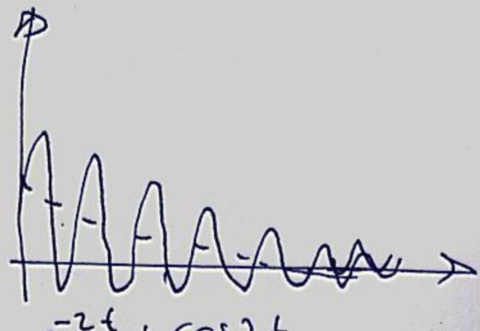
$e^{2t} \cos 3t$



$e^{-2t} \cos 3t$



$e^{2t} + \cos 3t$ $\quad t$



$e^{-2t} + \cos 3t$