E300 - NE213

1) Find the eigenvalues and eigenvectors of the following matrices

a) $A=\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$ b) $B=\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ c) $C=\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$

d) $D=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ e) $A=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

2) $\lambda_1 = 1$ and $\lambda_2 = 3$ are eigenvalues of the matrix $A=\begin{bmatrix} -8 & 1 & 6 \\ -11 & 4 & 6 \\ -13 & 1 & 10 \end{bmatrix}$ find λ_3.

3) $\lambda_1 = -2.6548 + 3.7195$ $\lambda_2 = 5.3138$ are the eigenvalues of the matrix $A=\begin{bmatrix} 1 & 0 & 3 \\ 5 & 0 & 0 \\ 0 & 6 & 2 \end{bmatrix}$ find λ_3

4) Prove that eigenvalues of a diagonal matrix are diagonal elements

5) prove that eigenvalues of an upper triangular matrix are diagonal elements

6) Find the eigenvalues of the following matri

a) $A=\begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$ b) $B=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 0 & 5 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

7) $\lambda_1 = 2$ $\lambda_2 = 4$ $\lambda_3 = 6$ are the eigenvalues of the matrix $A = \begin{bmatrix} -6 & -33 & 9 \\ 4 & 17 & -3 \\ 0 & 1 & 1 \end{bmatrix}$

Calculate the eigenvectors.

8) $\lambda_1 = 3$ $\lambda_2 = 2.824i$ $\lambda_3 = -2.824i$ are the eigenvalues of the matrix $A = \begin{bmatrix} 0 & 2 & -2 \\ 1 & 0 & 3 \\ 5 & 0 & 3 \end{bmatrix}$

Calculate the eigenvectors

9) The matrix A and its eigenvectors are given below.

$$A = \begin{bmatrix} 5 & -2 & -1 \\ -1 & 4 & -1 \\ 1 & -2 & 3 \end{bmatrix} \quad X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \quad X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Calculate eigenvalues.

10) A matrix is given as $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 4 & 2 \\ 1 & 1 & 5 \end{bmatrix}$

Determine which of the following vectors could be an eigenvector of A

a) $X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ b) $X_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ c) $X_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ d) $X_4 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ e) $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

11) Solve the following matrix differential Equation

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Selected answers

1) a) $A = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} \Rightarrow \begin{matrix} \lambda_1 = 4 \\ \lambda_2 = 6 \end{matrix}$ $x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

b) $B = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ $\begin{matrix} \lambda_1 = 1+2i \\ \lambda_2 = 1-2i \end{matrix}$ $x_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ $x_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

c) $C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ $\begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 0 \end{matrix}$ $x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

d) $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $\begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \end{matrix}$ $x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

2) $[A - \lambda I] = \begin{bmatrix} -8-\lambda & 1 & 6 \\ -11 & 4-\lambda & 6 \\ -13 & 1 & 10-\lambda \end{bmatrix}$ $\det[\ ] = \lambda^3 - 6\lambda^2 + 11\lambda - 6$

in a polynomial $x^n + a_{n-1}x^{n-1} + \dots a x + a_0 = 0$ the constant term is the multiplication of negative of the roots

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$ $(-\lambda_1)(-\lambda_2)(-\lambda_3) = -6$

$(-1)(\cancel{-3})(-\lambda_3) = -6$

$\boxed{\lambda_3 = 2}$

4) A diagonal matrix is

$A = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$ $[A - \lambda I] = \begin{bmatrix} a-\lambda & 0 & 0 & 0 \\ 0 & b-\lambda & 0 & 0 \\ 0 & 0 & c-\lambda & 0 \\ 0 & 0 & 0 & d-\lambda \end{bmatrix}$

$\det\{A - \lambda I\} = (a-\lambda)(b-\lambda)(c-\lambda)(d-\lambda) \cdots$

So the roots are

$\lambda_1 = a \qquad \lambda_2 = b \qquad \lambda_3 = c \qquad \lambda_4 = d$

5) Upper triangulor matrix is of the form

$$A = \begin{bmatrix} a & x & x & x \\ 0 & b & x & x \\ 0 & 0 & c & x \\ 0 & 0 & 0 & d \end{bmatrix} \qquad \{A - \lambda I\} = \begin{bmatrix} a-\lambda & x & x & x \\ 0 & b-\lambda & x & x \\ 0 & 0 & c-\lambda & x \\ 0 & 0 & 0 & d-\lambda \end{bmatrix}$$

it is known thot determinant of an upper diagonal matrix is multiplication of diagonal elements

$\det\{A - \lambda I\} = (a-\lambda)(b-\lambda)(c-\lambda)(d-\lambda) \ldots = 0$

Thus eigenvalues are $\lambda_1 = a \quad \lambda_2 = b \quad \lambda_3 = c \quad \lambda_4 = d$.

6) $A = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \qquad \begin{matrix} \lambda_1 = 8 \\ \lambda_2 = 9 \\ \lambda_3 = 3 \\ \lambda_4 = -2 \end{matrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 0 & 5 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 3 \\ \lambda_3 = 5 \\ \lambda_4 = 1 \end{matrix}$

7)

$$\begin{bmatrix} -6-2 & -33 & 9 \\ 4 & 17-2 & -3 \\ 0 & 1 & 1-2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} -8a - 33b + 9c = 0 \\ 4a + 15b - 3c = 0 \\ 0 \quad b \quad c = 0 \end{matrix}$$

Set $c = 1 \qquad b - c = 0 \Rightarrow b = 1$

$b = 1 \quad c = 1 \qquad 4a + 15b - 3c = 0 \Rightarrow a = \frac{-12}{4} = -3 \qquad X_1 = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} -6-4 & -33 & 9 \\ 4 & 17-4 & -3 \\ 0 & 1 & 1-4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad x_2 = \begin{bmatrix} -9 \\ 3 \\ 1 \end{bmatrix}$$

$$x_3 = \begin{bmatrix} -13 \\ 5 \\ 1 \end{bmatrix}$$

8)

$$\begin{bmatrix} -2.824i & 2 & -2 \\ 1 & -2.824i & 3 \\ 5 & 0 & 3-2.824i \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5a + (3 - 2.824i)c = 0$$

Set $c = 1$ $\qquad a = \dfrac{-3 + 2.824i}{5} = -0.6 + 0.564i$

$b = $ ---- —

$$x_1 = \begin{bmatrix} -0.6 + 0.564i \\ \text{----} \\ 1 \end{bmatrix}$$

$$A x_1 = \lambda_1 x_1$$

9) $$A = \begin{bmatrix} 5 & -2 & -1 \\ -1 & 4 & -1 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$5 + (-2)(-1) + (-1)1 = \lambda_1 \Rightarrow \lambda_1 = 2$$

$$A x_2 = \lambda_2 x_2$$

$$\begin{bmatrix} 5 & -2 & -1 \\ -1 & 4 & -1 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \lambda_2 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$5(-1) + (-2)(-1) + (-1)(1) = -\lambda_2$$

$$-4 = -\lambda_2$$

$$\boxed{4 = \lambda_2}$$

$$A x_3 = \lambda_3 x_3 \Rightarrow \underline{\lambda_3 = 6}$$

10)

$$\begin{bmatrix} 3 & 1 & -1 \\ 2 & 4 & 2 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

if $\lambda = 6$ equation is satisfied so $x_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector

$$\begin{bmatrix} 3 & 1 & -1 \\ 2 & 4 & 2 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

No λ satisfies this $x_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ is not eigenvector