

Existence of Solution in Linear Equations

Example LN1

$$2x+3y=4, \quad 3x+4y=5, \quad x+5y=9, \quad 6x+7y=8$$

$$\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 1 & 5 & 9 \\ 6 & 7 & 8 \end{bmatrix} \xrightarrow{\substack{-1.5R_1+R_2 \rightarrow R_2 \\ -0.5R_1+R_3 \rightarrow R_3 \\ -3R_1+R_4 \rightarrow R_4}} \begin{bmatrix} 2 & 3 & 4 \\ 0 & -0.5 & -1 \\ 0 & 3.5 & 7 \\ 0 & -2 & -4 \end{bmatrix}$$

$$\begin{matrix} 7R_2+R_3 \rightarrow R_3 \\ -4R_2+R_4 \rightarrow R_4 \end{matrix} \begin{bmatrix} 2 & 3 & 4 \\ 0 & -0.5 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 \\ 0 & -0.5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 2 & 3 & 4 \\ 0 & -0.5 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank } A = 2 \quad \text{rank } \bar{A} = 2 \quad n = 2$$

$$\text{rank } A = \text{rank } \bar{A} = n \rightarrow \text{Unique Solution}$$

Notice: Number of equations has no importance

Example LN2

$$2x+3y=4, \quad 3x+4y=5, \quad x+5y=9, \quad 6x+7y=2$$

$$\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 1 & 5 & 9 \\ 6 & 7 & 2 \end{bmatrix} \xrightarrow{\text{Row Operations}} \begin{bmatrix} 2 & 3 & 4 \\ 0 & -0.5 & -1 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 \\ 0 & -0.5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 2 & 3 & 4 \\ 0 & -0.5 & -1 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank } A = 2 \quad \text{rank } \bar{A} = 3 \quad n = 2$$

$$\text{rank } A \neq \text{rank } \bar{A} \rightarrow \text{No solution}$$

Example LN3

$$2x+3y=4, \quad 6x+9y=12, \quad 8+12y=16, \quad 4x+6y=8$$

$$\begin{bmatrix} 2 & 3 & 4 \\ 6 & 9 & 12 \\ 8 & 12 & 16 \\ 4 & 6 & 8 \end{bmatrix} \xrightarrow{\text{Row Operations}} \begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank } A = 1 \quad \text{rank } \bar{A} = 1 \quad n = 2$$

$$\text{rank } A = \text{rank } \bar{A} = r \quad r < n \rightarrow \text{Multiple Solution.}$$

$$n - r = 2 - 1 = 1. \quad 1 \text{ variable free}$$

Solution: Take the echelon form equation

$$2x + 3y = 4.$$

Set x freely and calculate y

$$\text{Set } x = 1 \rightarrow 2 + 3y = 4 \rightarrow y = 2/3 = 0.666$$

$$\text{Set } x = 0 \rightarrow y = 4/3 = 1.3333$$

$$\text{Set } x = -5.5 \rightarrow y = 5$$

Example LN4

$$2x+3y+4z-w=-5$$

$$6x+9y+12z-3w=21$$

$$\begin{bmatrix} 2 & 3 & 4 & -1 & -5 \\ 6 & 9 & 12 & -3 & 21 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 3 & 4 & -1 & -5 \\ 0 & 0 & 0 & 0 & 36 \end{bmatrix}$$

$$n = 4 \quad \text{rank } A = 1 \quad \text{rank } \bar{A} = 2$$

$$\text{rank } A \neq \text{rank } \bar{A} \rightarrow \text{No solution}$$

Example LN5

$$2x+3y+4z-w=-5$$

$$4x+2y+5z-3w=-3$$

$$\begin{bmatrix} 2 & 3 & 4 & -1 & -5 \\ 4 & 2 & 5 & -3 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 3 & 4 & -1 & -5 \\ 0 & -4 & -3 & -1 & 7 \end{bmatrix}$$

$$n = 4 \quad \text{rank } A = \text{rank } \bar{A} = r = 2$$

$$r < n \quad \text{multiple solution.}$$

$$n - r = 4 - 2 = 2 \quad 2 \text{ variable free}$$

To solve x and y , we use echelon form equations.

$$2x + 3y + 4z - w = -5$$

$$-4y - 3z - w = 7$$

$$\text{Set } z = 0 \quad w = 1 \quad (\text{arbitrary selection})$$

$$2x + 3y + 4 \cdot 0 - 1 = -5$$

$$-4y - 3 \cdot 0 - 1 = 7 \rightarrow y = -2$$

replace $y = -2$ $z = 0$ $w = 1$ into the first equation

$$2x + 3(-2) - 1 = -5 \rightarrow x = 1$$

$$\text{Solution 1} = [x = 1 \quad y = -2 \quad z = 0 \quad w = 1]$$

$$\text{Set } z = 2, \quad w = 3 \quad (\text{arbitrary selection})$$

$$2x + 3y + 4 \cdot 2 - 3 = -5 \rightarrow 2x + 3y = -10$$

$$-4y - 3 \cdot 2 - 3 = 7 \rightarrow -4y = 16 \rightarrow y = -4$$

from first equation

$$2x + 3y = -10 \rightarrow 2x + 3(-4) = -10 \rightarrow x = 1$$

$$\text{Solution 2} = [x = 1 \quad y = -4 \quad z = 2 \quad w = 3]$$

$$\text{Set } z = 0, \quad w = 0 \quad (\text{arbitrary selection})$$

$$2x + 3y + 4 \cdot 0 - 0 = -5 \rightarrow 2x + 3y = -5$$

$$-4y - 3 \cdot 0 - 0 = 7 \rightarrow -4y = 7 \rightarrow y = -1.75$$

from first equation

$$2x + 3y = -5 \rightarrow 2x + 3(-1.75) = -5 \rightarrow x = 0.125$$

$$\text{Solution 3} = [x = 0.125 \quad y = -1.75 \quad z = 0 \quad w = 0]$$

Example LN6

$$2x+3y+4z-w=-5, \quad 4x+2y+5z-3w=-3, \quad 6x+y+6z-5w=-1$$

$$-2x+5y+2z+3w=-9, \quad 10x+3y+11z-8w=-4$$

$$\begin{bmatrix} 2 & 3 & 4 & -1 & -5 \\ 4 & 2 & 5 & -3 & -3 \\ 6 & 1 & 6 & -5 & -1 \\ -2 & 5 & 2 & 3 & -9 \\ 10 & 3 & 11 & -8 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 3 & 4 & -1 & -5 \\ 0 & -4 & -3 & -1 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$n = 4 \quad \text{rank } A = \text{rank } \bar{A} = r = 2$$

$$r < n \quad \text{multiple solution.}$$

$$n - r = 4 - 2 = 2 \quad 2 \text{ variable free}$$

Echelon form equations are

$$2x + 3y + 4z - w = -5$$

$$-4y - 3z - w = 7$$

These equations are solved in **Example LN5**