Notes on matrix multiplication.

1) In general matrix multiplication is not commutative. $AB \neq BA$

2) if $A = 0$ $AB = 0'$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

3) if $B = 0$ $AB = 0$ $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

4) $AB = 0$ does not generaly imply $A = 0$ or $B = 0$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$A \neq 0$ $B \neq 0$ $AB = 0$

5) $AC = AD$ does not imply $C = D$

6) Theorem: if $AB = AC$ and A^{-1} exists then $B = C$

Proof
$$AB = AC$$
$$A^{-1}(AB) = A^{-1}(AC)$$
$$A^{-1}AB = A^{-1}AC$$
$$IB = IC$$
$$B = C$$

not $A^{-1}A = I$
$IB = BI = B$
$IC = CI = C$

7) if A^{-1} exists, and $AB = 0$ then $B = 0$

Proof
$$AB = 0$$
$$A^{-1}(AB) = A^{-1}[0]$$
$$A^{-1}AB = [0]$$
$$IB = 0$$
$$B = 0$$

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8) if A is singular so AB and BA is also singular

Proof

(a) if A is singular then A has dependent rows and dependent rows, and rank $A < n$
(n is the dimension of A)

(b)
$$A = \begin{bmatrix} x & y & z & w \end{bmatrix}$$ x, y, z, w are columns of

Since rows are linearly dependent then

$$ax + by + cz + dw = 0$$

or

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$ or $Ap = 0$ $p = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

c)
$$AP = 0$$
$$B\,AP = B.0$$
$$BAP = 0$$
$$[BA][P] = 0$$

The vector p is not zero vector and satisfies $[BA]\,P = 0$ thus columns of BA are linearly dependent.

So BA is singular

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EE-300

NA-101 N-1

1) The following vectors are given

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad y = \begin{bmatrix} 10 \\ 20 \\ 30 \\ 40 \end{bmatrix} \quad z = \begin{bmatrix} 10 \\ 20 \\ 30 \\ 100 \end{bmatrix}$$

show that x, y are Linearly dependent and x, z are Linearly independent.

Solution; method 1. Take x, y

$$ax + by = 0 \qquad a\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + b\begin{bmatrix} 10 \\ 20 \\ 30 \\ 40 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$a + 10b = 0 \rightarrow a = -10b$$
$$2a + 20b = 0 \rightarrow 2(-10b) + 20b = 0 \rightarrow 0 = 0$$
$$3a + 30b = 0 \rightarrow 3(-10b) + 30b = 0 \rightarrow 0 = 0$$
$$4a + 40b = 0 \rightarrow 4(-10b) + 40b = 0 \rightarrow 0 = 0$$

Result $\underline{a = -10b}$ is the solution

$$ax + by = 0 \rightarrow (-10b)x + by = 0 \rightarrow \underline{-10x + y = 0}$$

x and y are Linearly dependent

We can write $-10x + y = 0$ or $\underline{y = 10x}$

Now take x and z

$$ax + bz = 0 \qquad a\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + b\begin{bmatrix} 10 \\ 20 \\ 30 \\ 100 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$a + 10b = 0 \rightarrow a = -10b$$
$$2a + 20b = 0 \rightarrow 2(-10b) + 20b = 0 \rightarrow 0 = 0$$
$$3a + 30b = 0 \rightarrow 3(-10b) + 30b = 0 \rightarrow 0 = 0$$
$$4a + 100b = 0 \rightarrow 4(-10b) + 100b = 0 \rightarrow 60b = 0 \rightarrow \underline{b = 0}$$

$$a = -10b = -10 \times 0 = 0$$

$\boxed{a = 0, \ b = 0}$ Result x and z Lin. independent

<u>method 2 :</u> Take x and y

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 10 & 20 & 30 & 40 \end{bmatrix} \xrightarrow{-10R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There is a zero row x, y Linearly dependent

Now take x and z

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 10 & 20 & 30 & 100 \end{bmatrix} \xrightarrow{-10R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 60 \end{bmatrix}$$

No zero row $\Rightarrow$ x and z Linearly independent

<u>Method 3</u>

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad y = \begin{bmatrix} 10 \\ 20 \\ 30 \\ 40 \end{bmatrix}$$

it is clear that $y = -10x$

Thus x and y are Linearly dependent

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad z = \begin{bmatrix} 10 \\ 20 \\ 30 \\ 100 \end{bmatrix}$$

There is no α that satisfies $z = \alpha x$

thus z and x are Linearly independent

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N3

2) The following vectors are given

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad y = \begin{bmatrix} 10 \\ 20 \\ 30 \\ 40 \end{bmatrix} \quad z = \begin{bmatrix} 100 \\ 100 \\ 100 \\ 100 \end{bmatrix} \quad w = \begin{bmatrix} 101 \\ 102 \\ 103 \\ 104 \end{bmatrix}$$

a) x and y are Linearly dependent because

$\underline{y = 10x}$

b) x, z, w are Linearly dependent because

$w = z + x$

c) y, z, w are Linearly dependent because

$w = z + 0.1y$

d) x, z are Linearly independent because

no α satisfies $z = \alpha x$

e) x, w are Linearly independent

f) y, w " " "

g) z, w " " "

h) y, z " " "

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N4

3) The following vectors are given

$$x = \begin{bmatrix} 1 \\ -2 \\ 5 \\ 7 \end{bmatrix} \quad y = \begin{bmatrix} 3 \\ 2 \\ 4 \\ -1 \end{bmatrix} \quad z = \begin{bmatrix} 5 \\ -2 \\ 14 \\ 13 \end{bmatrix}$$

a) show that x, y are Linearly independent
b) show that x, z are Linearly independent
c) show that y, z are Linearly independent

d) show that x, y, z are Linearly dependent

Solution

a) $A = \begin{bmatrix} 1 & -2 & 5 & 7 \\ 3 & 2 & 4 & -1 \end{bmatrix} \xrightarrow{-3R_1 + R_2 \to R_2} \begin{bmatrix} 1 & -2 & 5 & 7 \\ 0 & 8 & -11 & -22 \end{bmatrix}$

Result : x, y are Linearly independent

b) $B = \begin{bmatrix} 1 & -2 & 5 & 7 \\ 5 & -2 & 14 & 13 \end{bmatrix} \xrightarrow{-5R_1 + R_2 \to R_2} \begin{bmatrix} 1 & -2 & 5 & 7 \\ 0 & 8 & -11 & -22 \end{bmatrix}$

x, z L. independent

c) $C = \begin{bmatrix} 3 & 2 & 4 & 1 \\ 5 & -2 & 14 & 13 \end{bmatrix} \xrightarrow{-\frac{5}{3}R_1 + R_2 \to R_2} \begin{bmatrix} 3 & 2 & 4 & 1 \\ 0 & -\frac{16}{3} & \frac{22}{3} & \frac{34}{3} \end{bmatrix}$

$y\ z$ Lin independent

d) $\begin{bmatrix} 1 & -2 & 5 & 7 \\ 3 & 2 & 4 & -1 \\ 5 & -2 & 14 & 13 \end{bmatrix} \begin{matrix} -3R_1 + R_2 \to R_2 \\ -5R_1 + R_3 \to R_3 \end{matrix} \begin{bmatrix} 1 & -2 & 5 & 7 \\ 0 & 8 & -11 & -22 \\ 0 & 8 & -11 & -22 \end{bmatrix}$

$-R_2 + R_3 \longrightarrow R_3 \quad \begin{bmatrix} 1 & -2 & 5 & 7 \\ 0 & 8 & -11 & -22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

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x, y, z are Linearly dependent.